Bound Entanglement Can Be Activated

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Bound entanglement is the noisy entanglement which cannot be distilled to a singlet form. Thus it cannot be used alone for quantum communication purposes. Here we show that, nevertheless, the bound entanglement can be, in a sense, pumped into a single pair of free entangled particles. It allows for teleportation via the pair with the fidelity impossible to achieve without support of bound entanglement. [S0031-9007(98)08152-6]

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Despite deep research, quantum entanglement still astonishes even specialists, producing highly nonintuitive effects such as quantum parallelism [1], quantum cryptography [2], quantum dense coding [3], quantum teleportation [4], and reduction of communication complexity [5]. In practice, one usually deals with noisy entanglement represented by *mixed* states of a composite system. The latter are entangled (inseparable) if they are not mixtures of product states [6,7]. However, the mixed state entanglement cannot be used directly for quantum communication purposes. The first example of procedure of distillation of this entanglement to useful singlet form has been provided by Bennett *et al.* [8] and discussed later in Ref. [9]. A similar procedure has been applied to quantum privacy amplification [10]. Subsequently, it has been shown [11] that the noisy entanglement of two spin-$\frac{1}{2}$ systems, however small, can be distilled to the singlet form. Then it was naturally supposed that the same is possible for larger systems. However, quite recently, it has been shown that beginning with two spin-1 systems, quantum mechanics implies existence of two qualitatively different kinds of noisy entanglement [12]: apart from the "free" entanglement which can always be distilled, there is a "bound" one which by no means can be brought to the singlet form. The curiosity of the bound entangled states is that to produce them one needs some amount of pure entanglement, while any, however little amount of it cannot be recovered back from them. The bound entanglement is closely connected with Peres separability criterion [13] (see also [14]). In particular, it has been shown [12] that if an inseparable state satisfies Peres criterion, then its entanglement is bound.

Existence of the bound entanglement involves new questions concerning local realism and quantum information. However, there is a question closely related to the practical topics. Namely, one can simply ask: Can the bound entanglement be somehow activated to produce *any* effect useful in quantum communication? In this paper we show that the bound entanglement can be, in a sense, liberated, giving, in particular, the possibility of improving transmission of quantum information. It also suggests the existence of new effects in a mixed state entanglement domain.

Before we present the main result of this paper let us recall that quite recently it has been pointed out [15,16] that mixed state free entanglement may have some disadvantage as it cannot be distilled *noncollectively*, i.e., by acting over single entangled pairs separately. It particular, it means that in some cases given a *single* pair of two spin-s particles in a free entangled (FE) state $\varrho_{\rm in}$ and using only quantum local operations (QL) and classical communication (CC), one cannot make the *fidelity* F of the resulting state $\varrho_{\rm out}$,

$$F(\varrho_{\rm out}) = \langle\Psi_+|\varrho_{\rm out}|\Psi_+\rangle,\qquad |\Psi_+\rangle = \frac{1}{\sqrt{2s+1}}\sum_{i=0}^{2s}|i\rangle|i\rangle \tag{1}$$

arbitrary close to 1. This is an important point as the fidelity (1) plays a central role in the teleportation scheme [4] if applied to mixed states [17].

Now, let us explain the main result of this paper. We consider just a *single* pair of spin-1 particles in a mixed state ϱ shared by spatially separated Alice and Bob who are allowed to make any QLCC operations. The state ϱ is taken to be FE, but its entanglement is so weak that in the case of a single pair, no QLCC operations can increase its fidelity upon some bound $C < 1$. We then introduce some new bound entangled (BE) states and show that if, in addition, Alice and Bob are provided with a large supply of pairs in those states, then they can skip the border C making now the fidelity of the original FE pair arbitrary close to 1 with nonzero probability. We shall hereafter call the process of making the fidelity F arbitrary close to unity *quasidistillation*, as, in contrast with the original distillation idea, we allow a number of initial pairs and a probability of success to depend on the required final F. The key point of the presented result is that the distinguished FE pair as well as the

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set of all BE pairs cannot be quasidistilled *themselves*. However, putting them *together* produces new quality from which the state with arbitrary good fidelity already can be obtained. The revealed process can be viewed as a kind of entanglement transfer from BE pairs into an FE pair. After presentation of the effect in detail we address the question of its possible relevance for quantum communication and show that a transfer of quantum information, which is impossible with the FE pair alone, can sometimes be done with the aid of a bound entanglement supply. Finally we discuss the possible relevance of the effect in the context of the original distillation idea. In particular, we conclude that our result suggests nonadditivity of the distillable entanglement [9].

To illustrate details of our scheme consider the case of two spin-1 particles. The state of any particle can be described by using three-dimensional Hilbert space spanned by basis states $|0\rangle, |1\rangle, |2\rangle$ corresponding to antiparallel, perpendicular, and parallel orientation of particle spin with respect to the z axis. This means, in particular, that we put $s = 1$ in formula (1).

For our purposes let us introduce mixed separable states,

$$\begin{aligned} \sigma_+ &= \tfrac{1}{3}(|0\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|2\rangle\langle 1|\langle 2| + |2\rangle|0\rangle\langle 2|\langle 0|), \\ \sigma_- &= \tfrac{1}{3}(|1\rangle|0\rangle\langle 1|\langle 0| + |2\rangle|1\rangle\langle 2|\langle 1| + |0\rangle|2\rangle\langle 0|\langle 2|). \end{aligned} \qquad (2)$$

Suppose now that Alice and Bob share a *single* pair of spin-1 particles in the following free entangled mixed state:

$$\varrho_{\text{free}} = \varrho(F) \equiv F|\Psi_+\rangle\langle\Psi_+| + (1 - F)\sigma_+, \quad 0 < F < 1. \qquad (3)$$

In fact, it is easy to see that the state is free entangled. Namely, after action of the local projections $(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$ we get an inseparable 2×2 state. The latter can always be distilled as shown in Ref. [11]. Hence, by definition, the state $\varrho(F)$ contains free entanglement. On the other hand, it can be shown [18] that the state can never be quasidistilled noncollectively, i.e., no QLCC performed on a single pair in the state $\varrho(F)$ can increase its fidelity upon some $C < 1$ (we do not present the proof here, as it is a rather technical task and requires a different approach than the ones applied so far).

Suppose, however, that apart from the pair in state $\varrho(F)$, Alice and Bob have a large number of pairs of particles in the following state [19]:

$$\sigma_\alpha = \frac{2}{7}|\Psi_+\rangle\langle\Psi_+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-. \qquad (4)$$

Those states admit simple characterization with respect to the parameter $2 \le \alpha \le 5$,

$$\sigma_\alpha \text{ is} \begin{cases} \text{separable for } 2 \le \alpha \le 3, \\ \text{bound entangled for } 3 < \alpha \le 4, \\ \text{free entangled for } 4 < \alpha \le 5. \end{cases} \qquad (5)$$

Let us briefly justify the above characterization. It is easy to point out separability of the states (4) for the first region $2 \le \alpha \le 3$. Indeed then σ_α can be written as a mixture of separable states (recall that separable states form the convex set) $\sigma_\alpha = \frac{6}{7}\varrho_1 + \frac{\alpha-2}{7}\sigma_+ + \frac{3-\alpha}{7}\sigma_-$. Here $\varrho_1 = (|\Psi_+\rangle\langle\Psi_+| + \sigma_+ + \sigma_-)/3$, which has been explicitly represented as a mixture of product states [20]. It is even more easy to find the free entanglement of σ_α in the last region $4 < \alpha \le 5$: it can be done in the same way as for the states (3). For the intermediate region $3 < \alpha \le 4$ direct calculation shows that σ_α satisfies Peres separability criterion [13]. Nevertheless, in this case the state is inseparable, and then, as such, it is bound entangled [12]. Here, instead of direct proving of this inseparability, we will show that such states can produce the effect which cannot come from any separable state. Namely, we shall show that if only Alice and Bob share a large number of pairs in the state σ_α with $3 < \alpha \le 4$ then they can quasidistill the state ϱ_{free}. Note that it would *not* be possible if the state σ_α were separable. Indeed, any usage of a separable state together with the QLCC action could be interpreted as some new QLCC action *alone*, since the separable state itself can be produced by means of some QLCC operation. However, as was mentioned before, *no* QLCC on a single pair in state (3) can quasidistill it. Thus, the possibility of quasidistillation of a single pair $\varrho(F)$ with help of the state σ_α with $3 < \alpha \le 4$ will be at the same time the proof that the latter is bound entangled. Note that any initial supply of BE states, if it represents the only entanglement in the process, cannot be quasidistilled [21].

Consider now the protocol of quasidistillation. Recall that Alice and Bob share one pair in the FE state (3) and a large supply of pairs in the BE states (4). They can proceed repeating the following two step procedure which is, in fact, a direct 3×3 analog of the one used in the distillation of entanglement [8,10,22]:

(i) They take the free entangled pair in the state $\varrho_{\text{free}}(F)$ and one of the pairs being in the state σ_α. They perform the bilateral XOR operation $U_{\text{BXOR}} \equiv U_{\text{XOR}} \otimes U_{\text{XOR}}$, each of them treating the member of a free (bound) entangled pair as a source (target). Recall here that the unitary XOR gate introduced in [8] and used in generalized form in [22,23] is defined as

$$U_{\text{XOR}}|a\rangle|b\rangle = |a\rangle|b \oplus a\rangle, \qquad b \oplus a = (b + a)\text{mod}N, \qquad (6)$$

where the initial state $|a\rangle$ ($|b\rangle$) corresponds to a source (target) state of N-level system.

(ii) Alice and Bob measure the z-axis spin components of the members of the target pair. Then they compare their results via phone. If the compared results differ from each other they have to discard both pairs and then the trial of improvement of F fails. If the results agree then the trial succeeds and they discard only the target pair, coming back with (as we shall see) an improved source pair to the first step (i).

By virtue of high symmetry of the states (3) and (4), it is easy to see that the success in step (ii) happens with nonzero probability,

$$P_{F\to F'} = \frac{2F + (1 - F)(5 - \alpha)}{7}, \quad (7)$$

leading then to the transformation $\varrho(F) \to \varrho(F')$ which produces fidelity

$$F'(F) = \frac{2F}{2F + (1 - F)(5 - \alpha)}. \quad (8)$$

If only α is greater than 3, the above continuous function of F exceeds the value of F on the whole region $(0, 1)$. Thus the successful repeating of steps (i) and (ii) produces the sequence of source fidelities $F_n \to 1$ (see Fig. 1). The probability of achieving any fidelity F_n is $P_n = (P_{F\to F'})^n$, hence, it is *nonzero* for any n. Thus all the states (4) with $3 < \alpha \leq 5$ allow us to quasidistill state (3). In particular, the effect holds for the region $3 < \alpha \leq 4$ confirming that the target state (4) is inseparable, hence bound entangled in this region. On the contrary for the region $2 \leq \alpha \leq 3$ the iteration of the formula (8) decreases fidelity (Fig. 1). This dramatic qualitative change reflects the fact that then Alice and Bob's large supply of pairs is in separable states which, as was indicated before, cannot help to quasidistill the pair in state (3). It is a remarkable result as it shows that the seemingly useless bound entanglement can be, in a sense, *pumped* into a single pair of free entangled particles. We expect a similar effect for other bound entangled states such as those introduced in Ref. [7].

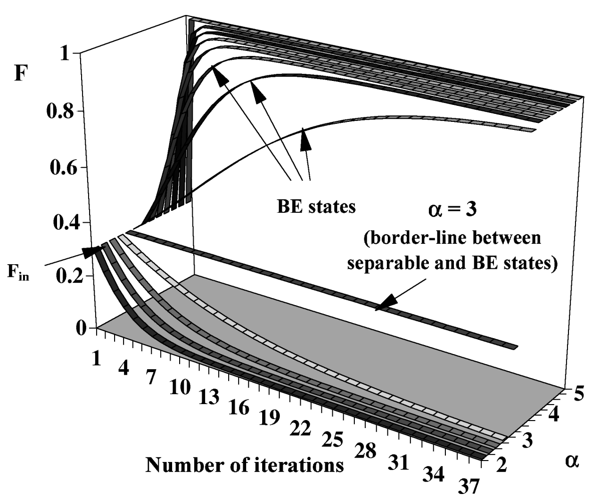

FIG. 1. Liberating bound entanglement. The singlet fraction of the FE state is plotted vs the number of successful iterations (i) and (ii) and the parameter α of the state ϱ_α of the used BE pairs. The initial singlet fraction of the FE pair is taken $F_{\text{in}} = 0.3$.

Let us discuss the physical meaning of the result. First we shall point out an interesting connection of the result with the special kind of quantum communication which is teleportation. Recall that any quantum state of the composite system ϱ can be regarded as a channel in the process of teleportation [17,18]. The idea is that Alice possesses one particle in an unknown state ψ and one member of the pair being in a state ϱ. Bob possesses another member of the pair. Then Alice and Bob perform some deliberately chosen QLCC operation and Bob finds his particle in the state resembling, at least to some degree, the initial unknown state ψ of Alice's particle. This is the most general quantum teleportation scheme. The fidelity of transmission of the state is measured by $f = \overline{\langle\psi|\Lambda_\varrho(\psi)|\psi\rangle}$ (call it transfer fidelity) where $\Lambda_\varrho(\psi)$ represents Bob's particle state after the whole procedure [17] and the bar stands for the average over all possible input states ψ. If the state ϱ, which forms the quantum teleportation channel is the maximally entangled state, then optimally chosen QLCC guarantees $\Lambda_\varrho(\psi) = |\psi\rangle\langle\psi|$ and the transfer fidelity f is equal to unity. However, in general, f can be less than 1 and the aim is to obtain the highest possible f. Alice and Bob can be interested in *conclusive teleportation* [24], where sometimes they obtain high fidelity, but sometimes decide not to teleport at all, discarding the pair they share. Then we can show how to apply the main result of this paper to the problem of quantum communication in the case of conclusive teleportation. It has been shown [18] that if Alice and Bob share only the single Fe pair in the state ϱ_{free} (3), then f is unavoidably bounded by some $f_{\max} < 1$. If, however, apart from the pair in the state ϱ_{free}, Alice and Bob share a lot of pairs in the BE state (4) then according to (i) and (ii) there is nonzero probability $p(F)$ that by using some QLCC operations they will achieve $F \geq f_{\max}$. If they succeed to achieve this then they can apply subsequently random unitaries of the form $U \otimes U^*$ (star denotes complex conjugation) and the standard teleportation scheme to obtain fidelity $f = (3F + 1)/4 \geq F$ (see Ref. [18]). All of the above is nothing but the scheme of conclusive teleportation which, by virtue of the used BE states, has the fidelity $f \geq F > f_{\max}$. As F can be made arbitrary close to 1 with nonzero probability, it is obvious that the support of the BE states allowed to skip the original border $f_{\max}$ of conclusive teleportation making it arbitrary close to 1.

A way of interpreting the results presented above is suggested by entanglement-energy analogy [12] (see also Ref. [25]). Namely, the situation is somewhat similar to the processes which need an initial supply of some amount of energy to be run. Here the role of the extra initial energy is played by the single free entangled pair, which allows us to run the process of drawing entanglement from the BE pairs.

In Ref. [12] the analogy entanglement energy was stated quantitatively, where the analog of useful (free) energy was the distillable entanglement D. Recall that distillable entanglement $D(\varrho)$ denotes the maximal number of singlet pairs per input pair which can be produced by means of QLCC operations from a large number of pairs in the state ϱ. Now we can expect an effect being in more strict analogy with the energy exchange processes. Namely, we expect that the distillable entanglement [9] $D(\varrho)$ may be *nonadditive*. In fact, by definition [12], for any FE state ϱ_{free} one has $D(\varrho_{\text{free}}) > 0$ while for BE ones $D(\varrho_{\text{bound}}) = 0$. Note that the presented quasidistillation scheme involves some kind of entanglement transfer from BE pairs into the FE one. It suggests that we may have $D(\varrho_{\text{free}} \otimes \varrho_{\text{bound}}) > D(\varrho_{\text{free}})$. But the latter is simply the sum $D(\varrho_{\text{free}}) + D(\varrho_{\text{bound}})$ as the last term vanishes by definition. This would really mimic a strange algebra in which $0 + 1$ would be greater than 1. Then the bound entanglement which is *not distillable at all if alone* could be distillable *through* free entanglement: the latter would be the window allowing to liberate the former. In terms of the mentioned analogy, the bound entanglement would perform for us useful informational work, if supported by, perhaps, a small supply of free entanglement. Then, the role of the latter would be to *activate* the bound one.

An even more probable effect strongly suggested by the present results is the following. Suppose that we enrich the actions Alice and Bob are conventionally allowed to do. Namely, apart from performing local quantum operations and classical communication, we allow them to share publicly any amount of bound entangled pairs. Now, what have shown in this paper is that the new class of operations (call it LQCC + BE) is significantly more powerful than the LQCC operations alone. Consequently, one expects that the distillable entanglement within this new paradigm can be strictly greater than the conventional one, i.e., we would have $D_{\text{LQCC+BE}} > D_{\text{LQCC}}$.

We would like to emphasize that the conjecture concerning the superadditivity of distillable entanglement can be formulated for quantum channel capacities [9]. Namely, consider a channel, which produces a BE state if one sends a half of singlet through it. We expect that the usage of that channel jointly with some other channel of nonzero capacity Q can result in a greater capacity than Q.

Finally, note that our discussion benefits from two opposite points of view. In one of them we treat the bound entanglement as a supplement which helps to handle with the free one. In the other one, the basis is bound entanglement, while the free one is only to activate it. We believe that both perspectives will be useful for further investigation of a role of the bound entanglement in quantum information theory.

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