Nonlinear Stability Theorem for High-Intensity Charged Particle Beams

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Global conservation constraints based on the nonlinear Vlasov-Maxwell equations are used to derive a three-dimensional kinetic stability theorem for an intense non-neutral ion beam (or charge bunch) propagating with average axial velocity $v_b = \text{const}$. It is shown that a sufficient condition for linear and nonlinear stability for perturbations with arbitrary polarization is that the equilibrium distribution be a monotonically decreasing function of the single-particle energy H' in the beam frame, i.e., $\partial f_{\text{eq}}(H')/\partial H' \leq 0$. [S0031-9007(98)06765-9]

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Periodic focusing accelerators [1–3] have a wide range of applications ranging from basic scientific research, to applications such as heavy ion fusion, tritium production, and spallation neutron sources. Of particular importance, at the high beam currents and charge densities of practical interest, are the effects of the intense self-fields produced by the beam space charge and current. While considerable progress can be made in understanding the evolution of the beam distribution function $f_b(\mathbf{x}, \mathbf{p}, t)$ and the average electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ in kinetic analyses [1,4–6] based on the nonlinear Vlasov-Maxwell equations, the effects of finite geometry and space charge often make predictions of detailed stability behavior difficult. It is therefore important to develop a basic understanding of the class of distribution functions that are *stable* and can propagate quiescently over large distances, even in parameter regimes where space-charge effects are intense and play a controlling role in the nonlinear beam dynamics. The present analysis makes use of global (spatially averaged) conservation constraints [6] satisfied by the nonlinear Vlasov-Maxwell equations to determine a sufficient condition for stability of an intense non-neutral ion beam (or isolated charge bunch) propagating in the positive z direction with average axial velocity $v_b = \text{const}$ along the axis of a perfectly conducting cylindrical pipe with wall radius $r = (x^2 + y^2)^{1/2} = r_w$. The theoretical approach used here is motivated by the early work [7] of Newcomb, Gardner, and Fowler, carried out for perturbations about a spatially uniform, electrically neutral, nonrelativistic plasma, and by the stability theorem developed by Davidson and Krall [6,8] for a one-component, nonrelativistic, non-neutral plasma column confined radially by a uniform axial magnetic field. The present analysis is restricted to a *collisionless* model of intense beam propagation based on the nonlinear Vlasov-Maxwell equations. That is, the beam particles interact with the average (applied plus collective) fields, but discrete-particle interactions (such as binary collisions) are *not* included in the model. The term *equilibrium,* as used in the present analysis, does not refer to thermal equilibrium, but rather to a quasisteady equilibrium state on a time scale short in comparison with a binary collision time.

The beam consists of positively charged ions with charge $+Z_i e$ and rest mass m propagating in the positive z direction with characteristic kinetic energy $(\gamma_b - 1)mc^2$ in the laboratory frame, where $\gamma_b = (1 - v_b^2/c^2)^{-1/2}$ is the relativistic mass factor. The particle motion in the beam frame (the "primed" frame) is assumed to be nonrelativistic with $|\mathbf{v}'| \ll c$, and the beam is assumed to have sufficiently high directed axial velocity that $v_b \gg |\mathbf{v}'|$. The beam current and charge density are allowed to be sufficiently intense that the collective processes associated with space-charge effects and self-consistent changes in the beam current can play a controlling role in the nonlinear evolution of the distribution of beam particles $f_b(\mathbf{x}, \mathbf{p}, t)$ in the six-dimensional phase space $(\mathbf{x}, \mathbf{p})$. Finally, it is assumed that transverse focusing of the beam particles is provided by the average effects of applied magnetic or electric focusing fields. We adopt a model widely used in the *smooth-focusing* approximation, which corresponds to a transverse focusing electric field of the form

$$\mathbf{E}^0_{sf}(\mathbf{x}) = -\frac{1}{Z_i e} m\omega^2_{\beta\perp}(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y), \qquad (1)$$

where $\omega_{\beta\perp} = \text{const}$ is an effective betatron frequency for transverse oscillations. Equation (1) is often used to model the *average* focusing properties of an alternating-gradient lattice of magnetic or electric quadrupoles.

Analysis of the nonlinear Vlasov-Maxwell equations is greatly simplified by transforming to a frame of reference moving at the average axial velocity $v_b = \text{const}$ of the beam particles, particularly because of the assumption that the particle motion is nonrelativistic in the beam frame. The Lorentz transformation relating the primed variables $(\mathbf{x}', \mathbf{p}', t')$ in the beam frame to the "unprimed" variables $(\mathbf{x}, \mathbf{p}, t)$ in the laboratory frame is given by

$$\begin{gathered} x' = x, \qquad y' = y, \qquad z' = \gamma_b(z - v_b t), \\ p'_x = p_x, \qquad p'_y = p_y, \\ p'_z = \gamma_b(p_z - \gamma m v_b), \qquad t' = \gamma_b(t - v_b z/c^2). \end{gathered} \qquad (2)$$

Here, the particle momentum and velocity are related by $\mathbf{p} = \gamma m\mathbf{v}$ and $\mathbf{p}' = \gamma' m\mathbf{v}'$, where the kinematic mass factors $\gamma = (1 + \mathbf{p}^2/m^2c^2)^{1/2}$ and $\gamma' = (1 + \mathbf{p}'^2/m^2c^2)^{1/2}$ transform according to $\gamma' = \gamma_b(\gamma - v_b p_z/mc^2)$. For the smooth-focusing electric field defined in Eq. (1), some straightforward algebra shows that the corresponding applied focusing force $\mathbf{F}'_{\text{foc}} = Z_i e(\mathbf{E}'_{\text{foc}} + c^{-1}\mathbf{v}' \times \mathbf{B}'_{\text{foc}})$ on a particle in the beam frame is given by

$$[\mathbf{F}'_{\text{foc}}]_{sf} = -\gamma_b m\omega^2_{\beta\perp}\bigg\{\bigg(1 + \frac{v'_z v_b}{c^2}\bigg)(x'\hat{\mathbf{e}}'_x + y'\hat{\mathbf{e}}'_y) - \frac{v_b}{c^2}(x'v'_x + y'v'_y)\hat{\mathbf{e}}'_z\bigg\}, \quad (3)$$

which can be approximated by

$$[\mathbf{F}'_{\text{foc}}]_{sf} = -\gamma_b m\omega^2_{\beta\perp}(x'\hat{\mathbf{e}}'_x + y'\hat{\mathbf{e}}'_y) \quad (4)$$

for $|\mathbf{v}'| \ll c$. Here, $(\hat{\mathbf{e}}'_x, \hat{\mathbf{e}}'_y, \hat{\mathbf{e}}'_z)$ are unit Cartesian vectors in the beam frame.

In the beam frame, the nonlinear Vlasov equation [1] for the distribution function $f_b(\mathbf{x}', \mathbf{p}', t')$ can be expressed as

$$\frac{\partial f_b}{\partial t'} + \frac{\partial}{\partial \mathbf{x}'}\cdot(\mathbf{v}' f_b) + \frac{\partial}{\partial \mathbf{p}'}\cdot\bigg\{\bigg[Z_i e\bigg(\mathbf{E}' + \frac{1}{c}\mathbf{v}' \times \mathbf{B}'\bigg) - \nabla'\psi'_{sf}\bigg]f_b\bigg\} = 0, \quad (5)$$

where $\mathbf{E}'(\mathbf{x}', t')$ and $\mathbf{B}'(\mathbf{x}', t')$ are the self-generated electric and magnetic fields, and the focusing force defined in Eq. (4) has been expressed as $[\mathbf{F}'_{\text{foc}}]_{sf} = -\nabla'\psi'_{sf}(\mathbf{x}')$, where

$$\psi'_{sf}(x', y') = \tfrac{1}{2}\gamma_b m\omega^2_{\beta\perp}(x'^2 + y'^2) \quad (6)$$

is the confining potential. Because the particle motion is assumed to be nonrelativistic in the beam frame, we approximate $\gamma' = 1 + \mathbf{p}'^2/2m^2c^2$ and $\mathbf{p}' = m\mathbf{v}'$. The static potential defined in Eq. (6) provides transverse confinement of the beam particles in the x'-y' plane, but not in the z' direction, which corresponds to a *continuous* beam. A simple generalization to the case of a single *finite-length charge bunch* is to add to Eq. (6) a stationary $(\partial/\partial t' = 0)$ contribution in the beam frame that provides axial confinement of the ions, e.g., a term proportional to $\gamma_b m\omega^2_{\beta z}z'^2/2$, where $\omega_{\beta z} = \text{const}$ is an effective betatron frequency for the axial motion. This gives the confining potential

$$\psi'_{sf}(x', y', z') = \tfrac{1}{2}\gamma_b m\omega^2_{\beta\perp}(x'^2 + y'^2) + \tfrac{1}{2}\gamma_b m\omega^2_{\beta z}z'^2. \quad (7)$$

It is convenient to view Eq. (7) as modeling the average potential of an rf bucket that provides a stationary confining potential centered at $z' = 0$ in the beam frame. The relative axial and transverse dimensions of the charge bunch confined by Eq. (7) will, of course, depend on the ratio $\omega_{\beta z}/\omega_{\beta\perp}$.

Maxwell's equations in the beam frame relate $\mathbf{E}'(\mathbf{x}', t')$ and $\mathbf{B}'(\mathbf{x}', t')$ self-consistently to the distribution function $f_b(\mathbf{x}', \mathbf{p}', t')$. We introduce the scalar and vector potentials, $\phi'(\mathbf{x}', t')$ and $\mathbf{A}'(\mathbf{x}', t')$, and express $\mathbf{B}' = \nabla' \times \mathbf{A}'$ and $\mathbf{E}' = \mathbf{E}'_L + \mathbf{E}'_T$, where $\mathbf{E}'_L = -\nabla'\phi'$ is the longitudinal electric field, $\mathbf{E}'_T = -c^{-1}\partial\mathbf{A}'/\partial t'$ is the transverse electric field, and the Coulomb gauge condition with $\nabla' \cdot \mathbf{A}' = 0$ is assumed. The Maxwell equations $\nabla' \cdot \mathbf{B}' = 0$ and $\nabla' \times \mathbf{E}' = -c^{-1}\partial\mathbf{B}'/\partial t'$ are automatically satisfied, and Poisson's equation and the $\nabla' \times \mathbf{B}'$ Maxwell equation are readily expressed in the beam frame as

$$\nabla'^2\phi' = -4\pi Z_i e\int d^3p' f_b, \quad (8)$$

$$\nabla'^2\mathbf{A}' = -\frac{4\pi}{c}Z_i e\int d^3p'\,\mathbf{v}' f_b + \frac{1}{c^2}\frac{\partial^2\mathbf{A}'}{\partial t'^2} + \frac{1}{c}\nabla'\frac{\partial\phi'}{\partial t'}, \quad (9)$$

where $\nabla'^2 \equiv \partial^2/\partial x'^2 + \partial^2/\partial y'^2 + \partial^2/\partial z'^2$. The electrostatic potential $\phi'(\mathbf{x}', t')$ is determined self-consistently in terms of the beam charge density by means of Eq. (8), and $\mathbf{A}'(\mathbf{x}', t')$ is determined in terms of the beam current density by means of Eq. (9). We impose the requirement that the tangential electric field and the normal magnetic field vanish at radius $r = r_w$. That is, $[E_z]_{r=r_w} = [E_\theta]_{r=r_w} = [B_r]_{r=r_w} = 0$, where B_r, E_θ, and E_z denote field components in cylindrical polar coordinates in the laboratory frame. In the beam frame, the corresponding field components are $E'_z = E_z$, $B'_r = \gamma_b(B_r + v_b E_\theta/c)$, and $E'_\theta = \gamma_b(E_\theta + v_b B_r/c)$, so that the corresponding boundary conditions in the beam frame are given by $[E'_z]_{r'=r_w} = [E'_\theta]_{r'=r_w} = [B'_r]_{r'=r_w} = 0$. In terms of the scalar and vector potentials, $\phi'(\mathbf{x}', t')$ and $\mathbf{A}'(\mathbf{x}', t')$, these boundary conditions can be expressed in the equivalent form

$$\phi'(r' = r_w, \theta', z', t') = A'_z(r' = r_w, \theta', z', t') = A'_\theta(r' = r_w, \theta', z', t') = 0, \quad (10)$$

where the constant values of ϕ', A'_z, and A'_θ at $r' = r_w$ have been set equal to zero.

The Vlasov-Maxwell equations (5), (8), and (9) provide a complete nonlinear description of the collective interaction of the beam particles with the applied and self-generated electric and magnetic fields. Equations (5), (8), and (9), subject to the boundary conditions in Eq. (10), can be used to derive certain *global* (spatially averaged) conservation constraints [6] in the beam frame that are useful in demonstrating a kinetic stability theorem. For present purposes, it is assumed that the phase-space density $f_b(\mathbf{x}', \mathbf{p}', t')$ is equal to zero beyond some radius r'_0, i.e., $f_b = 0$ for $r' = (x'^2 + y'^2)^{1/2} > r'_0 < r_w$. For the case of a charge bunch with finite axial length, i.e., when $\omega_{\beta z} \neq 0$ in Eq. (7), it is also assumed that $f_b = 0$ for $|z'| > L'_0/2$ where L'_0 is larger than the axial bunch length $2z'_b$. Moreover, when carrying out integrations over momentum $\mathbf{p}'$, it is assumed that $f_b = 0$ as $|\mathbf{p}'| \to \infty$. Finally, the domain of spatial integration is defined by $\int\frac{d^3x'}{L'}\cdots = \int_{-L'/2}^{L'/2}\frac{dz'}{L'}\int_0^{r_w}dr'r'\int_0^{2\pi}d\theta'\cdots$. Here, two cases are distinguished. Case (a) corresponds to an infinite-length beam where $\psi'_{sf}(x', y')$ is specified by Eq. (6), and L' is viewed as a fundamental periodicity length for Fourier decomposition of the z' dependence of the distribution function and field components. Case (b) corresponds to a finite-length charge bunch, where $\psi'_{sf}(x', y', z')$ is specified by Eq. (7) with $\omega_{\beta z} \neq 0$, and L' is chosen to be sufficiently large that $z' = L'/2$ and $z' = -L'/2$ are in the far-field regions of the charge bunch, where $\mathbf{E}'$ and $\mathbf{B}'$ are negligibly small.

The two global conservation constraints that are of particular utility correspond to the conservation of total particle plus field energy $U'(t')$, and generalized entropy $S'_G(t')$. Without presenting algebraic details [9], it can be shown from Eqs. (5), (8), (9), and (10) that

$$U'(t') = \frac{1}{L'} \int d^3x' \left\{ \frac{|\mathbf{E}'_T|^2 + |\mathbf{B}'|^2}{8\pi} + \int d^3p' \left(\frac{\mathbf{p}'^2}{2m} + \psi'_{sf} + \frac{1}{2} Z_i e \phi' \right) f_b \right\} = \text{const}, \tag{11}$$

and

$$S'_G(t') = \frac{1}{L'} \int d^3x' \int d^3p' \, G(f_b) = \text{const}, \tag{12}$$

no matter how complicated the nonlinear evolution of $f_b(\mathbf{x}', \mathbf{p}', t')$, $\mathbf{E}'(\mathbf{x}', t')$, and $\mathbf{B}'(\mathbf{x}', t')$. Here, $G(f_b)$ is a smooth, differentiable (but otherwise unspecified) function with $G(f_b \to 0) = 0$. In Eq. (11), $\mathbf{B}' = \nabla' \times \mathbf{A}'$ is the self-generated magnetic field, $\mathbf{E}'_T = -c^{-1}\partial \mathbf{A}'/\partial t'$ is the transverse electric field, and $\mathbf{E}'_L = -\nabla'\phi'$ is the longitudinal space-charge field. Use has been made of Eqs. (8) and (10) to express $(8\pi L')^{-1} \int d^3x' |\nabla'\phi'|^2 = (L')^{-1} \int d^3x' \int d^3p' (Z_i e/2) \phi' f_b$.

Equations (11) and (12) represent very powerful constraints on the nonlinear evolution of the system. A three-dimensional kinetic stability theorem can be derived by introducing a generalized Helmholtz free energy defined by $F'(t') = U'(t') + S'_G(t') = \text{const}$. We consider (arbitrary-amplitude) perturbations about a time-stationary ($\partial/\partial t' = 0$) equilibrium distribution $f_{\text{eq}}(\mathbf{x}', \mathbf{p}')$ in the beam frame and corresponding space-charge potential $\phi'_{\text{eq}}(\mathbf{x}')$. It is further assumed that the equilibrium distribution $f_{\text{eq}}(\mathbf{x}', \mathbf{p}')$ carries zero current in the beam frame, i.e., $\int d^3p' \mathbf{v}' f_{\text{eq}} = 0$, so that $\mathbf{A}'_{\text{eq}} = 0$ and $\mathbf{B}'_{\text{eq}} = 0 = \mathbf{E}'_{T_{\text{eq}}}$. Perturbed quantities are denoted by $\delta f_b(\mathbf{x}', \mathbf{p}', t') = f_b(\mathbf{x}', \mathbf{p}', t') - f_{\text{eq}}(\mathbf{x}', \mathbf{p}')$, $\delta\phi'(\mathbf{x}', t') = \phi'(\mathbf{x}', t') - \phi'_{\text{eq}}(\mathbf{x}')$, $\delta \mathbf{E}'_T(\mathbf{x}', t) = \mathbf{E}'_T(\mathbf{x}', t')$, and $\delta \mathbf{B}'(\mathbf{x}', t') = \mathbf{B}'(\mathbf{x}', t')$. Defining $\Delta F'(t') \equiv F'(t') - F'_{\text{eq}}$, where $F'(t') = U'(t') + S'_G(t')$, some straightforward algebra gives

$$\Delta F'(t') = \frac{1}{L'} \int d^3x' \left\{ \frac{|\delta \mathbf{E}'_T|^2 + |\delta \mathbf{B}'|^2 + |\nabla' \delta\phi'|^2}{8\pi} + \int d^3p' \left[\left(\frac{\mathbf{p}'^2}{2m} + \psi'_{sf} + Z_i e \phi'_{\text{eq}} \right) \delta f_b + G(f_{\text{eq}} + \delta f_b) - G(f_{\text{eq}}) \right] \right\} = \text{const}. \tag{13}$$

Here, $\delta\phi'(\mathbf{x}', t')$ and $\phi'_{\text{eq}}(\mathbf{x}')$ are related to $\delta f_b(\mathbf{x}', \mathbf{p}', t')$ and $f_{\text{eq}}(\mathbf{x}', \mathbf{p}')$ by

$$\nabla'^2 \delta\phi' = -4\pi Z_i e \int d^3p' \, \delta f_b, \qquad \nabla'^2 \phi'_{\text{eq}} = -4\pi Z_i e \int d^3p' \, f_{\text{eq}}, \tag{14}$$

and $\psi'_{sf}(\mathbf{x}')$ is defined in Eq. (7) for an axially confined charge bunch, and in Eq. (6) for a continuous beam. The coefficient of $\delta f_b(\mathbf{x}', \mathbf{p}', t')$ in Eq. (13) will be recognized as the Hamiltonian

$$H' = \frac{\mathbf{p}'^2}{2m} + \psi'_{sf}(\mathbf{x}') + Z_i e \phi'_{\text{eq}}(\mathbf{x}') \tag{15}$$

for single-particle motion in the combined applied focusing potential $\psi'_{sf}(\mathbf{x}')$ and equilibrium space-charge potential $\phi'_{\text{eq}}(\mathbf{x}')$.

A *linear* (small-signal) stability theorem can be obtained from Eq. (13) as follows. We Taylor expand $G(f_{\text{eq}} + \delta f_b) = G(f_{\text{eq}}) + G'(f_{\text{eq}})\delta f_b + G''(f_{\text{eq}})(\delta f_b)^2/2 + \cdots$, where $G'(f_{\text{eq}}) = \partial G(f_{\text{eq}})/\partial f_{\text{eq}}$, etc., and retain terms to quadratic order in perturbed quantities. This gives

$$[\Delta F']^{(2)} = \frac{1}{L'} \int d^3x' \left\{ \frac{|\delta \mathbf{E}'_T|^2 + |\delta \mathbf{B}'|^2 + |\nabla' \delta\phi|^2}{8\pi} + \int d^3p' \left[[H' + G'(f_{\text{eq}})](\delta f_b) + \frac{1}{2} G''(f_{\text{eq}})(\delta f_b)^2 + \cdots \right] \right\} = \text{const}. \tag{16}$$

We now choose $G(f_{\text{eq}})$, which has been arbitrary to this point, to satisfy $\partial G(f_{\text{eq}})/\partial f_{\text{eq}} = -H'$ so that the term linear in δf_b vanishes exactly in Eq. (13). This condition also gives $G''(f_{\text{eq}}) = -\partial H'/\partial f_{\text{eq}}$, so that Eq. (13) becomes (correct to second order)

$$[\Delta F']^{(2)} = \frac{1}{L'} \int d^3x' \left\{ \frac{|\delta \mathbf{E}'_t|^2 + |\delta \mathbf{B}'|^2 + |\nabla' \delta\phi|^2}{8\pi} + \frac{1}{2} \int d^3p' \frac{(\delta f_b)^2}{[-\partial f_{\text{eq}}/\partial H']} \right\} = \text{const}. \tag{17}$$

When $f_{\text{eq}}(\mathbf{x}', \mathbf{p}')$ depends on $(\mathbf{x}', \mathbf{p}')$ only through the Hamiltonian H', and when $f_{\text{eq}}(H')$ is a monotonically decreasing function of H' with

$$\frac{\partial}{\partial H'} f_{\rm eq}(H') \leq 0\,, \qquad (18)$$

it follows that the quantity $[\Delta F']^{(2)}$ defined in Eq. (17) is a sum of positive-definite terms. Therefore, because $[\Delta F']^{(2)} = $ const, no one of the terms in Eq. (17) can grow without bound, and we conclude that Eq. (18) is a *sufficient condition for linear stability* of the equilibrium $(f_{\rm eq}, \phi_{\rm eq})$ to small-amplitude perturbations δf_b, $\delta\phi'$, $\delta\mathbf{E}'_T$, and $\delta\mathbf{B}'$.

The exact global constraint condition (13) can be used to show that Eq. (18) is also a *sufficient condition for nonlinear stability* of the equilibrium to perturbations with arbitrary amplitude. Proof of this nonlinear stability theorem makes two successive applications of the mean-value theorem and proceeds as follows. The functional form of $G(f_b)$ in Eq. (13) is quite general. In the subsequent proof of the nonlinear stability theorem, we exploit this generality and further assume that $G(f_b)$ is a monotonically decreasing function of f_b and has positive concavity, i.e.,

$$\frac{\partial}{\partial f_b} G(f_b) \leq 0, \qquad \frac{\partial^2}{\partial f_b^2} G(f_b) \geq 0\,, \qquad (19)$$

over the entire range of values of the distribution function $f_b \geq 0$ accessible by the nonlinear Vlasov-Maxwell equations. Two successive applications of the mean-value theorem allows us to express the difference $G(f_{\rm eq} + \delta f_b) - G(f_{\rm eq})$ occurring in Eq. (13) in the form

$$G(f_{\rm eq} + \delta f_b) - G(f_{\rm eq}) = \frac{\partial G}{\partial f_b}\bigg|_{f_{\rm eq}+\delta f_{b1}} \delta f_b = \left[\frac{\partial G}{\partial f_b}\bigg|_{f_{\rm eq}} + \frac{\partial^2 G}{\partial f_b^2}\bigg|_{f_{\rm eq}+\delta f_{b2}} \delta f_{b1}\right]\delta f_b\,. \qquad (20)$$

Here, for positive perturbation $\delta f_b(\mathbf{x}', \mathbf{p}', t') \geq 0$, the quantities δf_{b1} and δf_{b2} lie in the intervals $0 \leq \delta f_{b2} \leq \delta f_{b1} \leq \delta f_b$, whereas for negative perturbation $\delta f_b \leq 0$, the quantities δf_{b1} and δf_{b2} lie in the intervals $\delta f_b \leq \delta f_{b1} \leq \delta f_{b2} \leq 0$. In either case, the product $\delta f_{b1}\delta f_b$ satisfies $\delta f_{b1}\delta f_b \geq 0$. We substitute Eq. (20) into Eq. (13) and eliminate the term linear in δf_b by choosing $[\partial G/\partial f_b]_{f_{\rm eq}} = -H'$, which also implies that $[\partial^2 G/\partial f_b^2]_{f_{\rm eq}} = -\partial H'/\partial f_{\rm eq}$. Equation (13) then becomes

$$\Delta F'(t') = \frac{1}{L'}\int d^3x' \left\{\frac{|\delta\mathbf{E}'_T|^2 + |\delta\mathbf{B}'|^2 + |\nabla'\delta\phi'|^2}{8\pi} + \int d^3p' \left(\frac{\partial^2 G}{\partial f_b^2}\bigg|_{f_{\rm eq}+\delta f_{b2}}\right)(\delta f_{b1}\delta f_b)\right\} = \text{const.} \qquad (21)$$

Because of the assumption $\partial^2 G/\partial f_b^2 \geq 0$ in Eq. (19), and because $\delta f_{b1}\delta f_b \geq 0$ follows by construction from the mean-value theorem, we conclude that the right-hand side of Eq. (21) is a sum of positive-definite terms, no one of which can grow without bound. Therefore, because $[\partial^2 G/\partial f_b^2]_{f_{\rm eq}} = -\partial H'/\partial f_{\rm eq} \geq 0$ by assumption, we conclude that $\partial f_{\rm eq}(H')/\partial H' \leq 0$ is a sufficient condition for nonlinear stability.

It is important to recognize the wide range of applicability of the three-dimensional stability theorem developed here. For example, there are many choices of distribution function $f_{\rm eq}(H')$ for which $\partial f_{\rm eq}/\partial H' \leq 0$, and the equilibrium is therefore stable. One such choice is the isotropic thermal equilibrium [1,8,10,11] distribution $f_{\rm eq} = g(H') \equiv \beta' \exp(-H'/T'_b)$, where β' and T'_b are positive constants. Most importantly, the stability theorem applies to perturbations about equilibria $f_{\rm eq}(H')$ with arbitrary polarization and initial amplitude; to *continuous beams* that are radially confined and infinite in axial extent ($\omega_{\beta\perp} \neq 0$, $\omega_{\beta z} = 0$); to *charge bunches* that are radially *and* axially confined ($\omega_{\beta\perp} \neq 0$ and $\omega_{\beta z} \neq 0$); and to beams with arbitrary space-charge intensity consistent with the requirement that the applied potential $\psi'_{sf}(\mathbf{x}')$ provide confinement of the beam particles. As a final point, it should be emphasized that the stability theorem has far wider applicability than to the case where $\psi'_{sf}(\mathbf{x}')$ has the simple quadratic dependence on x', y', and z' in Eq. (7), provided the confining potential is time stationary in the beam frame, i.e., $\partial\psi'_{sf}/\partial t' = 0$. The main requirement is that $\psi'_{sf}(\mathbf{x}')$ correspond to a *confining* potential, i.e., that the focusing force $[\mathbf{F}'_{\rm foc}]_{sf} = -\nabla'\psi'_{sf}$ is restoring.

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