rms Envelope Equations in the Presence of Space Charge and Dispersion

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The matching of charged particle beams into transport lines is usually done by using the rms envelope equations. The usual rms envelope equations, however, do not apply in the presence of bending magnets and a longitudinal momentum spread. A new set of equations is needed that simultaneously describes the rms envelopes of the beam and the dispersion function. A derivation is outlined in this paper. The new equations will make it possible to achieve proper matching of the rms envelopes and dispersion in the regime of highly space charge dominated beams. [S0031-9007(98)06476-X]

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A proper matching of the envelopes is usually necessary for a successful transport of space charge dominated beams. The matching forces the beam to undertake breathing modes consistent with the lattice periodicity and may prevent the onset of fast growing instabilities. To this purpose the rms envelope equations are currently used. The rms envelope equations were first derived in [1] by writing the equations giving the evolution of the moments of the beam distribution in phase space. In the general case one finds an infinite hierarchy of equations where equations for the lower moments involve moments of higher order. One can truncate the endless sequence and retain only the equations for the second moments if the rms emittance is known or it is an invariant. In a hydrodynamic approach this would be equivalent to assuming that the pressure or the temperature are known or constant. For a generic Hamiltonian system the rms emittance is an exact invariant only in the linear approximation. However, in some cases it turns out to be roughly preserved even in the presence of nonlinear forces. Such an approximate invariance makes the rms envelope equations a useful and widely used tool for matching. In addition, a proper rms matching itself can reduce the possibility of an increase in the emittance. However, it should be pointed out that even for a perfectly matched beam a number of phenomena can still cause emittance growth. Among these are the usually fast instabilities driven by possible resonances between the envelope breathing and the motion of the individual particles [2]; the formation of a halo [3]; the equipartioning between the longitudinal and transverse degrees of freedom [4].

One limitation of the usual rms envelope equations is that they apply only to straight transport lines or monochromatic beams. The problem is that the rms emittance is no longer a linear invariant in the presence of the coupling between the longitudinal momentum and the betatron motion that occurs in the bending magnets. In some cases the use of the standard rms envelope equations may still be adequate even if bending magnets are present, provided that the beam current is not too high. However, the growing interest for space-charge dominated beams in spallation neutron sources, heavy ion inertial fusion, advanced high-energy colliders (e.g., muon colliders) small recirculators, beam cooling, and other applications is pushing toward achieving a tune depression in a range of values where proper matching can be obtained only with more accurate tools. This is the case for the Maryland Electron Ring [5] where a tune depression as small as 0.2 is planned to be reached.

The aim of this paper is to present a new set of rms envelope equations for continuous beams that can be applied to the general case where bends and a longitudinal momentum spread are present. The first step is to find a linear invariant that replaces the usual rms emittance in the more general case. Not surprisingly, the new rms invariant turns out to be dependent on the dispersion function. Since, on the other hand, the dispersion function must depend on the space charge forces and therefore on the envelopes of the beam, one realizes that a consistent treatment needs to involve both the equation for the envelopes and the dispersion function. The derivation of the new set of equations for the case of a continuous beam makes use of the following two results proved in [1] for a generic beam density distribution displaying an elliptic symmetry: (i) the second moments of the beam distribution depend only on the linear part of the forces, and (ii) the linear part of the self-force defined in an rms sense depends only on the second moments (envelopes) of the distribution. Earlier attempts to incorporate dispersion into the framework of the rms envelope equations [2,6] are not satisfactory since they are based on the assumption of the linear invariance of the standard rms emittance. We should mention that different approaches also employing moment equations for space-charge dominated beams have been introduced in [7] to study the effect of longitudinal momentum spread in bends.

Finally, we show a simple application of the new equations by evaluating the transition of a continuous beam from a straight transport line into a small recirculator in

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the smooth approximation. The main purpose is to offer a test of the new equations against a self-consistent calculation as presented in [8]. The results from this particular example also show that for a tune depression smaller than 0.5 the new equations provide predictions that deviate from the ones obtained from the usual rms envelope equations.

Consider beam of charged particles of mass m in a transport line with local radius of curvature $\rho(z)$, subject to a linear focusing (no space-charge forces for the moment) on the horizontal plane and having a longitudinal momentum $p_z = p_o(1 + \delta)$ with a relative deviation δ from the design momentum p_o, with E_o being the corresponding energy. We assume that there is no longitudinal focusing. Such a system is described by the Hamiltonian

$$H = \frac{1}{2} p_x^2 + \frac{k_x(z)}{2} x^2 + \frac{m^2 c^4}{E_o^2} \delta^2 - \frac{x}{\rho(z)} \delta . \quad (1)$$

One can easily verify that because of the coupling term $\delta x/\rho$ the standard rms emittance $\epsilon_x^2 = (\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2)$, is not an invariant for the system (1). Here $\langle \cdot \rangle$ denotes the averaging over the phase space variables. Our problem is to find a new quantity similar to the rms emittance that is preserved in the presence of dispersion. The strategy is to write a canonical transformation that removes the coupling and casts the Hamiltonian in a form for which the invariant can be immediately written. The invariant in the original variables is then recovered by reversion of the canonical transformation.

A suitable canonical transformation is generated by $G_2(x, \overline{p}_x, z) = \overline{p}_x[x - \delta D(z)] + x \delta D'(z)$, where $D(z)$ is a function that will eventually be identified with the dispersion function: $\overline{x} = x - \delta D(z)$ and $p_x = \overline{p}_x + \delta D'(z)$.

The transformed Hamiltonian reads

$$\overline{H} = \frac{1}{2} \overline{p}_x^2 + \frac{k_x}{2} \overline{x}^2 + \frac{m^2 c^4}{E_o^2} \delta^2 + \delta \overline{x}\left(D'' + k_x D - \frac{1}{\rho}\right) + \dots .$$

The remaining terms $(\dots)$ in the Hamiltonian contain a combination of δ and functions of z. Since they affect only the evolution of the variable canonically conjugated to δ (i.e., the RF phase), which we are not interested in, they can be disregarded. Clearly, if the function D is chosen in such a way that $D'' + k_x(z) D - 1/\rho(z) = 0$, the coupling between δ and $\overline{x}$ vanishes and we can immediately conclude that the quantity $\overline{\epsilon}_x^2 = (\langle \overline{x}^2 \rangle \langle \overline{p}_x^2 \rangle - \langle \overline{x}\overline{p}_x \rangle^2)$ is a linear invariant for the Hamiltonian $\overline{H}$. Then, if we express $\overline{\epsilon}_x$ in terms of the old variables we obtain a quantity invariant for our original system. We have

$$\begin{aligned} \langle \overline{x}^2 \rangle &= \int \overline{x}^2 f(\overline{x}, \overline{p}_x, \delta)\, d\overline{\mu} \\ &= \int (x - \delta D)^2 f(\overline{x}(x, p_x, \delta), \overline{p}_x(x, p_x, \delta), \delta)\, d\mu \\ &= \langle x^2 \rangle - 2D\langle x\delta \rangle + D^2 \langle \delta^2 \rangle . \end{aligned} \quad (2)$$

Here, $d\mu = dx\, dp_x\, d\delta$, $d\overline{\mu} = d\overline{x}\, d\overline{p}_x\, d\delta$, and $d\mu = d\overline{\mu}$ because the transformation is canonical. Notice that since $f(\overline{x}, \overline{p}_x, \delta)$ is a generic function, then $\tilde{f}(x, p_x, \delta) = f[\overline{x}(x, p_x, \delta), \overline{p}_x(x, p_x, \delta), \delta]$ is also a generic function. That is, the equality (2) holds for any distribution function. By the same token one can evaluate $\langle \overline{p}_x^2 \rangle$ and $\langle \overline{x}\overline{p}_x \rangle$ in terms of the moments in the old variables, and finally write the new invariant as

$$\begin{aligned} \epsilon_{\overline{x}}^2 = \epsilon_{dx}^2 &= (\langle x^2 \rangle - 2D\langle x\delta \rangle + D^2 \langle \delta^2 \rangle) \\ &\times (\langle p_x^2 \rangle - 2D' \langle p_x \delta \rangle + D'^2 \langle \delta^2 \rangle) \\ &- (\langle x p_x \rangle - D\langle p_x \delta \rangle - D' \langle x\delta \rangle + DD' \langle \delta^2 \rangle)^2 . \end{aligned} \quad (3)$$

We have introduced the notation ϵ_{dx} for the new invariant, to which in the following we will refer as "generalized emittance." Notice that in a straight beam line, where no dispersion is present ($D = D' = 0$) or for a monochromatic beam with vanishing longitudinal momentum spread ($\langle \delta^2 \rangle = 0$), the quantity (3) coincides with the usual rms emittance. Incidentally, we notice that the invariant introduced here is different from the generalized emittances defined, e.g., in [9]. Those quantities are examples of "kinematic invariants" the invariance of which is preserved under any linear symplectic map. The quantity defined here is more similar to a "dynamic invariant" (see [9] for this terminology), the form of which depends on the details of the physical system through the dispersion function $D(z)$.

Let us now introduce the space-charge forces into the picture. The Hamiltonian describing a continuous beam with space charge and in the presence of dispersion reads (q is the charge, v_z the longitudinal velocity, γ the relativistic factor)

$$\begin{aligned} H = &\frac{1}{2} (p_x^2 + p_y^2) + \frac{k_x(z)}{2} x^2 + \frac{k_y(z)}{2} y^2 \\ &+ \frac{q}{m v_z^2 \gamma^3} \psi(x, y, z) - \frac{\delta}{\rho(z)} x + \frac{m^2 c^4}{E_o^2} \delta^2 . \end{aligned} \quad (4)$$

The self-force is described by the potential ψ, and includes the contribution from both the magnetic and electric field (see, e.g., [4]). In writing (4) we assumed that the nonlinearities due to the external focusing and the transverse current due to the bending of the beam are negligible. From the Vlasov equation associated with the Hamiltonian (4) one can derive the following equations for the second moments ($g_o = q/m v_z^2 \gamma^3$):

$$\frac{d}{dz} \langle x^2 \rangle = 2 \langle x p_x \rangle , \quad (5a)$$

$$\frac{d}{dz} \langle p_x^2 \rangle = -2k_x \langle x p_x \rangle - 2g_o \left\langle p_x \frac{\partial}{\partial x} \psi \right\rangle + \frac{2}{\rho} \langle p_x \delta \rangle , \quad (5b)$$

$$\frac{d}{dz} \langle x p_x \rangle = \langle p_x^2 \rangle - k_x \langle x^2 \rangle - g_o \left\langle x \frac{\partial}{\partial x} \psi \right\rangle + \frac{1}{\rho} \langle x\delta \rangle , \quad (5c)$$

$$\frac{d}{dz}\langle x\delta\rangle = \langle p_x\delta\rangle, \tag{6a}$$

$$\frac{d}{dz}\langle p_x\delta\rangle = -k_x\langle x\delta\rangle - g_o\left\langle\delta\frac{\partial}{\partial x}\psi\right\rangle + \frac{1}{\rho}\langle\delta^2\rangle. \tag{6b}$$

The equations for $\langle y^2\rangle$, $\langle p_y^2\rangle$, $\langle yp_y\rangle$ are similar to those for $\langle x^2\rangle$, $\langle p_x^2\rangle$, $\langle xp_x\rangle$ when the dispersive term in the Hamiltonian is absent. Under the assumption of a beam with elliptic symmetry one can prove [1] that the terms in Eqs. (5), (6) involving the self-potential can be derived by an effective potential of the form

$$\frac{q}{mv_z^2\gamma^3}\psi(x,y,z) = -\frac{K}{4(\sigma_x+\sigma_y)}\left(\frac{x^2}{\sigma_x}+\frac{y^2}{\sigma_y}\right), \tag{7}$$

where K is the perveance [4] and $\sigma_x = \sqrt{\langle x^2\rangle}$ and $\sigma_y = \sqrt{\langle y^2\rangle}$. Given a self-potential of the form (7) the resulting equation for the dispersion function is

$$D'' + \left[k_x(z) - \frac{K}{2\sigma_x(\sigma_x+\sigma_y)}\right]D = \frac{1}{\rho(z)}. \tag{8}$$

Use of Eq. (7) to write the equation for the dispersion function accounts to assuming a linear approximation, with the linear part of the force due to space charge defined in an rms sense (as shown in [1]). To the extent that such an approximation holds, the generalized rms emittance ϵ_{dx} defined in (3) is invariant. One can simplify the expression for the generalized rms emittance by observing that the two equations (6) can be combined into a single equation that has the same form as (8). This allows us to identify $\langle x\delta\rangle = \langle\delta^2\rangle D(z)$ and $\langle p_x\delta\rangle = \langle\delta^2\rangle D(z)'$. Consequently,

$$\epsilon_{dx}^2 = (\langle x^2\rangle - D^2\langle\delta^2\rangle)(\langle p_x^2\rangle - D'^2\langle\delta^2\rangle) - (\langle xp_x\rangle - DD'\langle\delta^2\rangle)^2. \tag{9}$$

Next, we can use ϵ_{dx} to express $\langle p_x^2\rangle$ in terms of the other moments. By doing so, one can rewrite the rms equations as

$$\sigma_x'' = \frac{\epsilon_{dx}^2 + (\sigma_x\sigma_x' - DD'\langle\delta^2\rangle)^2}{\sigma_x(\sigma_x^2 - D^2\langle\delta^2\rangle)} - \frac{1}{\sigma_x}(\sigma_x')^2 - k_x\sigma_x + \frac{K}{2(\sigma_x+\sigma_y)} + \frac{\langle\delta^2\rangle}{\sigma_x}\left(\frac{D}{\rho}+D'^2\right), \tag{10}$$

$$\sigma_y'' = \frac{\epsilon_y^2}{\sigma_y^3} - k_y\sigma_y + \frac{K}{2(\sigma_x+\sigma_y)}. \tag{11}$$

The three equations (8),(10),(11), in the variables D, σ_x, σ_y, provide a consistent description for the evolution of the rms envelopes of a beam in a dispersive channel. They can be used to achieve a simultaneous matching of the rms beam envelopes and dispersion function. We emphasize the fact that the invariant ϵ_{ds} appearing in Eq. (10) coincides with the rms emittance at those locations where $D = 0$. Finally, notice that for $1/\rho(z) = 0$ or $\langle\delta^2\rangle = 0$, we recover the usual envelope equations for straight transport lines.

As an application and a test of the theory outlined in this paper we now want to estimate the variation of the beam parameters in the transition from a straight beam line into a small ring. In the model we consider both the focusing functions and the radius of curvature in the circular channel are z independent (smooth approximation). Furthermore, the external focusing is the same in the straight and in the circular channel. We assume that the beam undergoes a transition between two stationary distributions before and after injection. For the purposes of this calculation we do not need to specify how this matched injection can be achieved. In general the solution of the matching problem would require the solution of the rms envelope-dispersion differential equations (8),(10),(11) for specified initial and final beam conditions. The stationary solution before injection reads

$$\frac{(\epsilon_x^2)_s}{\sigma_{xs}^3} - k_x\sigma_{xs} + \frac{K}{2(\sigma_{xs}+\sigma_{ys})} = 0 \tag{12}$$

(similar equation for σ_{ys} with σ_{xs} and σ_{ys} interchanged). The subscript s indicates that the various quantities refer to the beam in the straight channel before injection. For a given value of the beam perveance K and emittance the equation above can be solved for the rms size of the beam σ_{xs}, σ_{ys}. After injection in the circular channel a stationary beam must satisfy the set of equations (8),(10),(11) with $\sigma_y' = \sigma_x' = D' = 0$. The connection between the two sets of rms quantities is provided by the assumption that the generalized emittance (3) in the horizontal plane is conserved through injection. Of course, the vertical rms emittance is also preserved because dispersion in our model affects the motion only in the horizontal plane. We have calculated a numerical solution for this set of equations for various values of the perveance or beam current for the particular case of an initially round beam $\sigma_{xs} = \sigma_{ys} = 0.5$ cm. The other parameters are [10] $k_x = k_y = 17.44$ m, $\rho = 1.82$ m, corresponding to an undepressed rms tune of $\nu_{ox} = \nu_{oy} = 7.6$. $\sqrt{\langle\delta^2\rangle} = 0.007$. For any given value of the perveance, the emittance $(\epsilon_x)_s = (\epsilon_y)_s$ is tuned in such a way that different beams with different perveance have the same rms radius at injection.

In Fig. 1 we plot the rms horizontal size of the beam after injection as a function of the tune depression. The value is scaled with respect to the rms horizontal size before injection. The tune depression ν/ν_o (i.e., the ratio between the rms tune in the presence of space charge ν, and the tune in the absence of space charge ν_o) is also calculated with respect to the beam before injection. We observe that the effect of higher space charge is to enlarge the beam horizontally. The curve is compared to results obtained in [8] where the same system was studied by looking for self-consistent solutions of the Vlasov-Poisson equations in the form of generalized KV beams [11] in a recirculator (dots in the picture), with all the nonlinearity due to space charge taken into account. The agreement corroborates the validity of the new set of

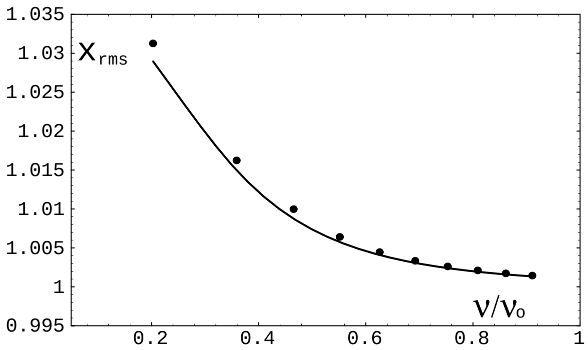

FIG. 1. Scaled horizontal rms beam size $x_{\rm rms} = \sigma_x/\sigma_{xs}$, after injection into a small ring as a function of the tune depression ν/ν_o (solid line). The curve is compared with the results (dots) obtained in [8] by studying self-consistent beam distributions in the ring satisfying the Vlasov-Poisson equation.

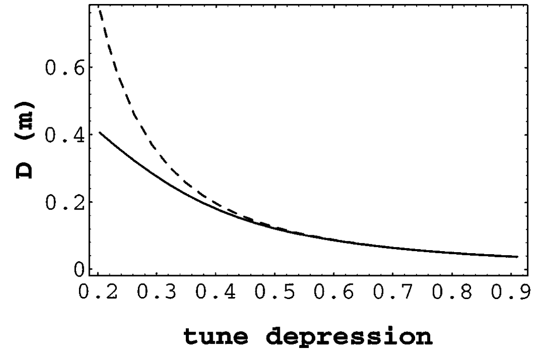

FIG. 2. Dispersion function in a small ring as a function of the tune depression ν/ν_o (solid line). The dashed line is the value of the dispersion function as calculated without taking into account the enlargement of the horizontal beam size in the ring due to dispersion.

equations. The relative growth of the horizontal beam envelope for this particular example is modest. However, if present, a mismatch in the dispersion could introduce a breathing in the horizontal beam size having a relative amplitude a few times larger than the values shown in the picture (see [8]). Moreover, a weaker external focusing could cause the relative growth to be considerably higher. Associated with the beam size growth is also a relative increase in the rms emittance (for this example it is about 15% at $\nu/\nu_o = 0.2$). Again, a dispersion mismatch would introduce oscillations in the rms emittance with a relative amplitude a few times larger (which could eventually turn into an irreversible emittance growth in the relaxation toward equilibrium).

In Fig. 2 we plot the value of the dispersion function. Since the smooth approximation has been assumed in this simple model, the dispersion function is constant along the ring. The result is compared (dashed line) to the dispersion function that we would obtain if we did not take into account the enlargement of the beam described in Fig. 1. One would get this result (dashed line) if the standard rms envelope equations were to be used in combination with Eq. (8). As anticipated, the two curves begin to get separated at $\nu/\nu_o \simeq 0.5$. Such an effect should, for example, be taken into account for a proper correction of chromaticity.

In conclusion, the goal of this paper was to present the derivation of a set of equations that generalize the standard rms envelope equations to the case where dispersion is present. The new set is made of three equations as opposed to the two equations for the vertical and horizontal rms envelopes of the beam in the usual approach, with the extra equation needed to accommodate the additional variable of the problem, i.e., the dispersion function. We expect the new equations will have useful applications in very high current regimes whenever proper envelope and dispersion matching is needed or an accurate knowledge of dispersion is required for the purpose of chromaticity correction.

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[11] The generalized Kapchinsky-Vladimirsky (KV) beam distribution as defined in [8] has a KV-like distribution in the transverse variables (delta function of the transverse Hamiltonian) and a Gaussian distribution in the longitudinal momentum spread. Because of dispersion the forces due to the space charge are nonlinear.