Stability, Multistability, and Wobbling of Optical Gap Solitons

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The stability of optical gap solitons is investigated in the framework of the generalized massive Thirring model. The self-phase modulation can induce either oscillatory or nonoscillatory types of instability, the latter mechanism being described by an explicit criterion, and unfolded geometrically to reveal gap soliton multistability, as well as different nonlinear evolution scenarios. [S0031-9007(98)06464-3]

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One of the most intriguing manifestations of nonlinearity is the energy localization at frequencies such that the propagation is otherwise forbidden, owing to periodicity-induced gaps in the linear dispersion relation. This energy trapping is mediated by *lattice* or *grating self-transparent* solitary waves, or simply *gap solitons* (GS), which hence constitute universal modes of periodic nonlinear media [1–9]. Although this concept holds in different contexts such as photon-atom coupling [3], electrical circuits [4], solid-state lattice vibrations [5], and photonics band gap materials [6], the field of nonlinear optics [1,2,7–10] is particularly attractive since a mature technology of grating fabrication makes the observation of GS viable with different nonlinearities. Fiber gratings offer unsurpassed opportunities to observe GS sustained by cubic nonlinearities [7], whereas bichromatic parametric GS [8,9] constitute a formidable laboratory for quadratic solitons [10]. A stunning feature of optical GS not yet observed is their unique capability to "freeze" the electromagnetic energy at zero (or extremely low) velocity in the laboratory frame. However, the *stability* of GS, which is a fundamental prerequisite for their observability, was not assessed yet, except for a few numerical experiments [2].

The stability of solitary waves is usually brought back to the parameter dependence of their conservation laws, as well known, e.g., for generalized nonlinear Schrödinger equations (NLSEs, i.e., dispersion originating from a Laplacian) [11]. In spite of recent extensions of such criteria for NLSEs [12–15] and noteworthy attempts to develop general theories [16–18], a complete understanding of solitons with *multiple parameters and invariants*, such as GS, is still lacking. Here, we predict different instability mechanisms for bright GS. Exponential instabilities are described by a novel *explicit criterion*, whose geometrical interpretation reveals for the first time GS *multistability*, and lead to distinct nonlinear evolution scenarios. Because of *oscillatory* instabilities that we also predict (e.g., for fibers [7]), not envisaged for other optical solitons [11–15], this criterion turns out to be only a sufficient one for instability. Specifically, the coupled-mode approach for optical GS yields Hamiltonian equations for forward (+) and backward (−) dimensionless envelopes $u_\pm$ at Bragg (gap-center) carrier frequency [2,9]

$$i\partial_t u_\pm = \frac{\delta H}{\delta u_\pm^*} = \mp i\partial_z u_\pm - u_\mp - \frac{\partial f_{nl}}{\partial u_\pm^*}, \quad (1)$$

where $H = \mathrm{Re}[\langle u_-, i\partial_z u_-\rangle_c - \langle u_+, i\partial_z u_+\rangle_c - 2\langle u_+, u_-\rangle_c] - \int dz\, f_{nl}$, and $f_{nl} = f_{nl}(u_\pm, u_\pm^*)$ accounts for the nonlinearity [19]. Although our approach can be easily extended to parametric GS (i.e., multicomponent $u_\pm$ [9]), and general nonlinearities f_{nl}, here we focus for definiteness on the generalized Thirring model such that $f_{nl} = \rho|u_+u_-|^2 + \frac{\sigma}{2}(|u_+|^4 + |u_-|^4)$ [2]. The invariance of Eqs. (1) under the transformation $u_\pm, (z,t,\rho,\sigma) \to \pm u_\pm, -(z,t,\rho,\sigma)$, allow us to take without loss of generality $\rho = 1$ letting either $\sigma > 0$ (e.g., the fiber case [7]), or $\sigma < 0$ (e.g., Kerr limit in quadratic media [8]). Solitary waves of Eqs. (1) are known to exist for $\sigma > 0$ [2], whereas Lorentz invariance and integrability by inverse scattering holds only for $\sigma = 0$ [20]. Setting $u_\pm = u_\pm^r + iu_\pm^i$ we rewrite Eqs. (1) in terms of the element $u = [u_+^r, u_-^r, u_+^i, u_-^i]^T$ of a real Hilbert space, as

$$\partial_t u = -\Omega H', \quad (2)$$

where $\Omega_{11} = \Omega_{22} = 0$ and $\Omega_{21} = -\Omega_{12} = I$ are elements of the simplectic operator $\Omega = -\Omega^{-1}$ (0 and I are 2×2 null and unit, respectively), and the prime denotes henceforth the Frèchet derivative $\frac{\delta}{\delta u}$. Other conservations of Eqs. (2) correspond to invariance $H(u) = H(Tu)$ of H under the action of a unitary group T, and are found as $\langle -\Omega^{-1}T'u, u\rangle$, where T' is the infinitesimal operator [16]. The rotational $[T_\Delta = \exp(-\Delta t\Omega), T'_\Delta = -\Omega]$, and translational $[T_v = \exp(-vt\partial_z),\ T'_v = -\partial_z]$ symmetries of Eqs. (2), where v and Δ are the symmetry parameters yield the conserved photon flux Q and momentum M

$$Q(u) = \frac{1}{2}\langle u, u\rangle; \qquad M(u) = \frac{1}{2}\langle u, -\Omega\partial_z u\rangle. \quad (3)$$

Let us consider solutions of Eqs. (2) in the form $u(z,t) = T_\Delta T_v \varphi(z,t) = \exp(-\Delta t\Omega - vt\partial_z)\varphi(z,t)$. It follows from Eqs. (2) that φ obeys the equation

$$\partial_t\varphi = -\Omega F'; \qquad F = F(\varphi) \equiv H - \Delta Q - vM, \quad (4)$$

where F plays the role of a Lyapunov functional. Solitary waves are by definition two-parameter stationary ($\partial_t = 0$) solutions $\varphi = \varphi_0$ of Eq. (4). Hence they are extrema $[F'(\varphi_0) = 0]$ of F, representing moving envelopes with frequency detuning Δ (from gap center) measured in the reference frame traveling with soliton velocity v. Here $F'(\varphi_0) = 0$ admits analytical solutions recast as

$$u_{\pm,0} = C_{\pm}\left(\frac{1 \pm v}{1 \mp v}\right)^{1/4} \sqrt{\eta(2a\zeta)}\, e^{i[\beta\zeta - \Delta t + \phi_{\pm}(a\zeta)]}, \quad (5)$$

where $\zeta \equiv \gamma(z - vt - z_0)$, z_0 is arbitrary, $\beta \equiv \gamma v \Delta$ is the propagation constant, $a \equiv \sqrt{1 - \delta^2}$ is the inverse width, $\delta \equiv \gamma\Delta$, and $\gamma \equiv (1 - v^2)^{-1/2}$ is the Lorentz factor. The intensity profile is $\eta(x) = 2a^2/[\cosh(x) + s\delta]$, the nonlinear phases $\phi_{\pm}(x) = \pm sF_{\pm} \tan^{-1}[\sqrt{(1 - s\delta)/(1 + s\delta)}\tanh(x)]$, and the constants are $C_+ = (\gamma^2|\chi|)^{-1/2}$, $C_- = -sC_+$, with $\chi = (1 - v^2) + \sigma(1 + v^2)$, $s = \mathrm{sgn}(\chi)$, and $F_+ = \frac{3R-1}{1+R}$, $F_- = \frac{3-R}{1+R}$, with $R = [1 + \sigma\gamma^2(1 + v)^2]/[1 + \sigma\gamma^2(1 - v)^2]$. Below we make use of the soliton invariants calculated from Eqs. (3)–(5): $Q_0 = Q(\varphi_0) = 4C_+^2 \tan[a^{-1}(1 - s\delta)]$, and $M_0 = M(\varphi_0) = \gamma^3 C_+^2 v\{2aC_+^2[(5 + v^2)\sigma + \gamma^{-2}] - 4s\delta\sigma Q_0\}$. These GS exist within the "dynamical" (i.e., measured in the soliton frame) frequency gap $\Delta^2 < 1 - v^2$, where linear solutions are exponentially decaying [1,2,9]. Inside the gap, whenever $-1 < \sigma < 0$ the GS (5) become singular ($\chi \to 0$, $C_{\pm} \to \infty$) as $v^2 \to v_s^2 = \frac{1+\sigma}{1-\sigma}$.

Linearization around the soliton, i.e., $\varphi = \varphi_0 + \delta\varphi$, yields the evolution equation $\partial_t \delta\varphi = -\Omega L \delta\varphi$ where $L \equiv F''(\varphi_0)$ is a self-adjoint 4×4 first-order differential operator. Exponentially growing perturbations [$\delta\varphi \sim \exp(\lambda t)$, $\mathrm{Re}(\lambda) > 0$] fulfill the standard form eigenvalue problem with the nonself-adjoint operator $-\Omega L$

$$-\Omega L \delta\varphi = \lambda\delta\varphi; \qquad L = H''(\varphi_0) + v\Omega\partial_z - \Delta I\,. \quad (6)$$

Here L has an unbounded negative and positive continuum which makes it difficult to apply a rigorous derivation (e.g., using spectral decomposition of L [16,17]) of an invariant criterion for the appearance of unstable eigenvalues of $-\Omega L$. We adopt here a multiscale approach valid under the assumption of adiabatic evolution [13]. Let us define slow time scales $t_n = \epsilon^n t$ ($\partial_t = \partial_0 + \epsilon\partial_1 + \epsilon^2\partial_2 + \ldots$, and denote $\partial_n f$ as f_{t_n}), and expand $\varphi = \varphi_0 + \delta\varphi = \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2 + \ldots$, assuming that the parameters $p \equiv (\Delta, v)$ change over the slow time scales so that $p = p(t_1, t_2, ...)$. Then we collect terms of the same order in ϵ after substitution in Eq. (4).

At zero order, we retrace the stationary equation $F'(\varphi_0) = 0$. At order ϵ, we obtain the linear problem $L\varphi_1 = S$ where $S \equiv \Omega(\Delta_{t_1}\partial_\Delta\varphi_0 + v_{t_1}\partial_v\varphi_0)$ does not depend explicitly on φ_1. This equation can be solved when S does not belong to the kernel of L, say $\mathrm{Ker}(L)$ i.e., rotational and translational modes associated with the symmetries: $\Omega\varphi_0$ and $\partial_z\varphi_0$, respectively. The orthogonality constraint $\langle S, \mathrm{Ker}(L)\rangle = 0$ yields two equations, in turn implying $(Q_0)_{t_1} = (M_0)_{t_1} = 0$, i.e., constancy of Q_0, M_0 over the slow time scale (i.e., adiabatic evolution). These equations can also be recast in the form of the following 2×2 system with unknown Δ_{t_1}, v_{t_1}

$$J(\varphi_0)\begin{bmatrix} \Delta_{t_1} \\ v_{t_1} \end{bmatrix} = 0; \qquad J(\varphi) \equiv \begin{pmatrix} \partial_\Delta Q & \partial_v Q \\ \partial_\Delta M & \partial_v M \end{pmatrix}. \quad (7)$$

The solvability $J \equiv \det J(\varphi_0) = 0$ of Eqs. (7) determines the threshold for the appearance of unstable modes, thereby defining the marginal stability condition

$$J = \partial_\Delta Q_0 \partial_v M_0 - (\partial_v Q_0)^2 = 0\,, \quad (8)$$

where we exploited the identity $\partial_v Q = \partial_\Delta M$. $J = 0$ has an explicit but cumbersome expression as $f(\Delta, v) = 0$. To proceed and characterize stable and unstable domains we substitute the asymptotic expansion $\varphi = \varphi_0 + \delta\varphi$ into F calculated over the perturbed function φ, and Taylor expanded around the soliton value $F_0 = F(\varphi_0)$, i.e., $F(\varphi) = F_0 + \langle F'(\varphi_0), \delta\varphi\rangle + \frac{1}{2}\langle F''(\varphi_0)\delta\varphi, \delta\varphi\rangle + \ldots$. At order ϵ^2, we obtain [exploiting the relations $F'(\varphi_0) = 0$ and $\langle F''(\varphi_0)\delta\varphi, \delta\varphi\rangle = \langle L\varphi_1, \varphi_1\rangle$]

$$F - F_0 = \frac{m_{\Delta\Delta}}{2}\dot{\Delta}^2 + \frac{m_{vv}}{2}\dot{v}^2 + m_{\Delta v}\dot{\Delta}\dot{v}\,, \quad (9)$$

where the dot stands for derivative with respect to the fast time t, and $m_{ij} = \langle\Omega\partial_i\varphi_0, L^{-1}\Omega\partial_j\varphi_0\rangle$, with $i, j = \Delta, v$. The 2 degrees of freedom oscillator (9) can be further reduced by exploiting the relation $\dot{\Delta} = -\frac{\partial_v Q}{\partial_\Delta Q}\dot{v} = -\frac{\partial_\Delta M}{\partial_v M}\dot{v}$ imposed by Eq. (7) close to the threshold where the adiabatic approach holds true. Choosing v as variable (an equivalent picture holds for Δ), Eq. (9) yields the energy $E \equiv -H(\varphi)$ of a unidimensional oscillator governing the evolution of the perturbed GS

$$E = U + \frac{m_v}{2}\dot{v}^2, \quad (10)$$

where $U \equiv \Delta(Q_0 - Q) + v(M_0 - M) - H_0$, and $m_v = 2m_{\Delta v}\frac{\partial_v Q_0}{\partial_\Delta Q_0} - m_{vv} - m_{\Delta\Delta}(\frac{\partial_v Q_0}{\partial_\Delta Q_0})^2$, are equivalent potential and mass, respectively. From Eqs. (9) and (10), for small variations of the parameters $\Delta(t) = \Delta_0 + \epsilon\Delta_\epsilon q(t)$, $v(t) = v_0 + \epsilon v_\epsilon q(t)$, we obtain the linearized oscillator $\ddot{q} = -\omega_0^2 q$, where $\omega_0^2 \equiv \frac{J}{m_v\partial_\Delta Q_0}$. For $|v| < |v_s|$, since $m_v\partial_\Delta Q_0$ is positive, we conclude that $J > 0$ corresponds to stability ($\omega_0^2 > 0$, bounded motion), whereas $J < 0$ yields instability ($\omega_0^2 < 0$, unbounded motion), and vice versa for $|v| > |v_s|$. The inspection of the condition (8) reveals that a change of stability occurs whenever $\sigma < 0$. The unstable domain ($J < 0$) reduces progressively as $|\sigma|$ decreases, and for $\sigma \geq 0$ only the stable region ($J > 0$) survives. In terms of

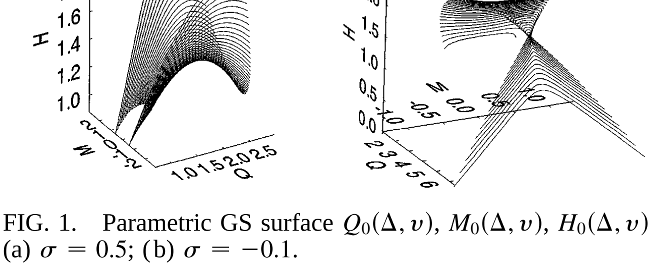

FIG. 1. Parametric GS surface $Q_0(\Delta, v)$, $M_0(\Delta, v)$, $H_0(\Delta, v)$: (a) $\sigma = 0.5$; (b) $\sigma = -0.1$.

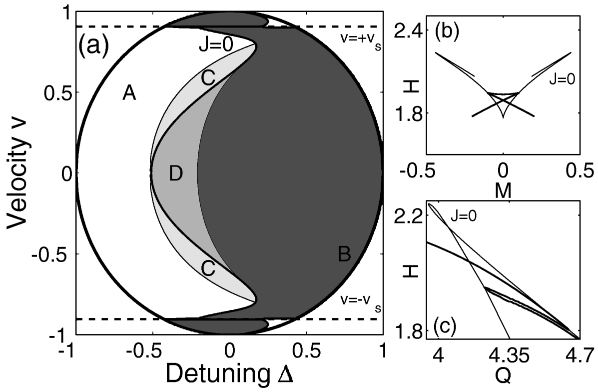

FIG. 2. (a) GS stability domains within the gap $\Delta^2 + v^2 < 1$ (see text). Along the dashed lines $v = \pm v_s$, GS (5) become singular. The insets show the GS surface (Q_0, M_0, H_0) projected on the planes (b) $Q = 4.4$ and (c) $M = 0.01$ (thick lines). Its singular edges bound the range of multistability and are delimited by the marginal curve $J = 0$ (thin line).

the eigenvalue problem (6) the marginal condition $J = 0$ describes pure *exponential* instabilities, i.e., real eigenvalue pairs bifurcating from the origin. However, in principle GS can also undergo *oscillatory* instabilities associated with complex (allowed by nonadjointness) eigenvalues of Eqs. (6) with $\mathrm{Re}(\lambda) > 0$. Such mechanism, known in other contexts, cannot be related to an invariant description [17], and to the best of our knowledge has not been reported for the linearization (6) arising for other lowest-order optical solitary waves (e.g., nonintegrable NLSEs [12–15]). Our extensive numerical solutions of Eq. (6) show that complex eigenvalues originating from the collision of two imaginary eigenvalues destabilize the GS for $\sigma > 0$, indicating that the marginal criterion (7) is only sufficient for instability. For $\sigma = 0.5$ [2,7], unstable complex λ are found in the lower half gap ($\Delta < 0$): e.g., stationary ($v = 0$) GS destabilize below the critical value $\Delta \simeq 0$.

Let us return to the $J = 0$ instability transition for $\sigma < 0$. An alternative approach, namely, the analysis of the parametric soliton surface $Q_0(\Delta, v), M_0(\Delta, v), H_0(\Delta, v)$, permits us to disclose the mechanisms which underlie this transition. The mapping $(\Delta, v) \to (Q_0, M_0, H_0)$ defines an associated Jacobian matrix $A = A(\varphi_0)$, with

$$A(\varphi) \equiv \begin{pmatrix} \partial_\Delta Q & \partial_\Delta M & \partial_\Delta H \\ \partial_v Q & \partial_v M & \partial_v H \end{pmatrix}. \tag{11}$$

The transition is accompanied by a nontrivial folding of this surface whose singular points are such that $\mathrm{rank}[A(\varphi_0)] = 1$. A necessary and sufficient condition for this to occur is given by the threshold condition $J = 0$. In fact, the third column in $A(\varphi_0)$ is a linear combination of the first two columns due to the two relations $\partial_p H_0 - \Delta \partial_p Q_0 - v \partial_p M_0 = 0$, $p = (\Delta, v)$, which follow from $\frac{dF}{dp} = \langle F', \varphi_p \rangle + \frac{\partial F}{\partial p}$ particularized to the soliton $[F'(\varphi_0) = 0]$. To show the difference between the full focusing and the defocusing cases, we show in Fig. 1 the GS invariant surface: when $\sigma > 0$ it is a single-value smooth function of the parameters [Fig. 1(a)], whereas for $\sigma < 0$ a singular folding of the mapping [catastrophe [18,21]] takes place [Fig. 1(b)]. As a consequence, soliton multistability occurs, i.e., GS solutions with different values of the parameters Δ, v, and Hamiltonian H exist for the same given values of M and Q (see also Fig. 2). This proves soliton multistability to be a general phenomenon not restricted to non-Kerr nonlinearities and NLSEs as in Refs. [22]. For a given σ, the multistable region in the plane (Δ, v) can be delimited geometrically searching for multiple intersections of $Q, M =$ const curves. The results are summarized in Fig. 2(a): within the gap (i.e., existence domain $\Delta^2 + v^2 < 1$), four bounded regions correspond to qualitatively different solutions: unstable (A) and stable (B) single-value GS; unstable (C) and stable (D) branches of multistable GS. As shown in Figs. 2(b) and 2(c), the projections of the surface Q_0, M_0, H_0 (calculated over a sufficiently wide range, e.g., $|v|, |\Delta| < 0.7$) on the planes $Q, M =$ const confirm the coexistence of three branches (one stable and two unstable), with typical shapes of a swallow-tail catastrophe [Fig. 2(b)] [21]. The marginal curve $J = 0$ always separates stable and unstable solutions in the parameter plane [Fig. 2(a)], and is confirmed [see Figs. 2(b) and 2(c)] to describe the singular edges of the GS surface (Q_0, M_0, H_0).

The stability results reported so far have been confirmed by extensive numerical integration of Eqs. (1). The decay typical of oscillatory instabilities is shown in Fig. 3, for $\sigma > 0$. The exponential growth rate and the oscillation period in Fig. 3(b) agree well with λ_r and $2\pi/\lambda_i$, respectively, $\lambda = \lambda_r + i\lambda_i$ being the unstable eigenvalue [inset of Fig. 3(b)] found numerically from Eq. (6).

For $\sigma < 0$ the evolution of unstable GS is more complicated: close to threshold, our analysis reveals that

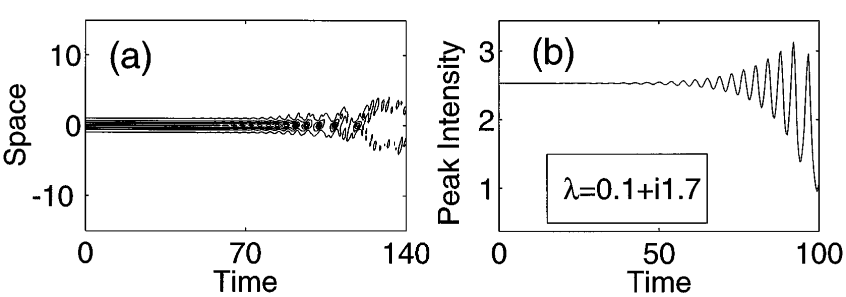

FIG. 3. Oscillatory GS instability for $\sigma = +0.5$, $\Delta = -0.9$, $v = 0$: (a) Spatio-temporal contour of intensity $|u_+|^2$; (b) Evolution of peak intensity $|u_+(0)|^2$. The inset shows the unstable eigenvalue λ found numerically from Eq. (6).

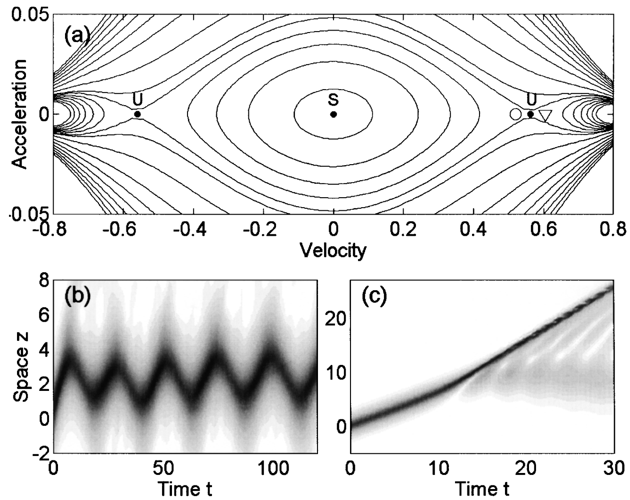

FIG. 4. (a) Phase-plane $(v, \dot{v})$ from Eq. (10), $\sigma = -0.1$. The fixed points (full circles) represent one stable (S) and two unstable (U) GS. Evolutions (the grey scale is proportional to $|u_+|^2$); (b) Soliton wobbling around the stable soliton S from input with $v_i = 0.54$, $\Delta(v_i) = -0.18$, circle in (a); (c) soliton decay into a foward linear wave traveling with $v = 1$ from input with $v_i = 0.6$, $\Delta(v_i) = -0.12$, triangle in (a).

unstable neighboring GS evolve in two qualitatively distinct ways. Consider perturbed inputs $u_i = u(t = 0)$ in Eqs. (1) which correspond to given values $Q = Q_i$ and $M = M_i$, and examine the nonlinear evolutions described by Eq. (10). Close to the threshold $J = 0$, the curves $Q_i =$ const approach the curves $M_i =$ const, and we can calculate the mass m_v and the potential $U = U(v)$ from the parametric dependence $\Delta = \Delta(v)$ calculated along the curve $Q = Q_i$ (or $M = M_i$). The level curves of E in the phase plane $(v, \dot{v})$ are shown in Fig. 4(a) for $Q_i = 4.4$ and $M_i = 0$ (for $M_i \neq 0$ the curves are no longer symmetric). The elliptic (S) and hyperbolic (U) points in Fig. 3(a) represent stable and unstable branches of multistable GS such that $(Q_0, M_0) = (Q_i, M_i)$. Moreover, the open curves accumulate due to divergence of m_v (cf. relativistic oscillator) as the singularity $v = \pm v_s = \pm 0.9$ is approached (i.e., limit of validity of nonlinear model). Now consider a generic initial condition on this phase plane $(v, \dot{v}) = (v_i, 0)$, describing a soliton with velocity v_i perturbed to have flux Q_i and momentum M_i. Two different types of motion are expected depending on v_i: wobbling and decay correspond to closed (around S) and open orbits, respectively. Integration of Eq. (1) confirms such expectations. In Fig. 4(b) we have chosen v_i to yield an oscillatory orbit. As predicted the perturbed unstable GS gives rise to long-range oscillations around a soliton with zero mean velocity (dynamical GS multistability) or, in other words, a wobbling soliton, which experience also a coupled periodic change of frequency. Velocity and frequency changes induce out of phase amplitude oscillation of u^+ and u^-, and soliton breathing (i.e., width and in-phase amplitude changes), respectively. In Fig. 4(c) v_i corresponds to an open orbit: the perturbation induces the velocity to increase monotonically. This process is accompanied by radiative decay of one GS component (also beyond the adiabatic regime), until the natural group velocity $|v| = 1$ is reached leading to a linear forward wave (backward for $v_i < 0$).

In summary, we have predicted different instability mechanisms for GS. An invariant criterion associated with the appearance of real eigenvalues can be interpreted geometrically to reveal multistability. Contrary to other optical solitons [11–14] also oscillatory instability takes place for GS. Our approach and results could be generalized to other physical situations (e.g., parametric walking solitons [15], codirectional coupling [23], different nonlinearities) modeled by nonintegrable models which lack Galilean or Lorentz invariance.

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