Kinetic Ising Model in an Oscillating Field: Finite-Size Scaling at the Dynamic Phase Transition

S. W. Sides,[1,2] P. A. Rikvold,[1,2] and M. A. Novotny[2]

[1]*Center for Materials Research and Technology and Department of Physics, Florida State University, Tallahassee, Florida 32306-4350*

[2]*Supercomputer Computations Research Institute, Florida State University, Tallahassee, Florida 32306-4130*

We study hysteresis for a two-dimensional spin-$1/2$ nearest-neighbor kinetic Ising ferromagnet in an oscillating field using Monte Carlo simulations. The period-averaged magnetization is the order parameter for a proposed dynamic phase transition (DPT). To quantify the nature of this transition, we present the first finite-size scaling study of the DPT for this model. Evidence of a diverging correlation length is given, and we provide estimates of the transition frequency and the critical indices β, γ, and ν.
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Although nonequilibrium phase transitions have been studied for over two decades, the understanding of their universality and scaling properties remains much weaker than for equilibrium critical phenomena. An important tool to study equilibrium phase transitions is finite-size scaling, which has also been applied to nonequilibrium transitions in which both the driving force and the different states are stationary, and the nonequilibrium behavior results from the dynamical rules. Examples include diffusive lattice gas models [1,2], Ising models with competing kinetic processes at different temperatures [3], and systems with multiplicative noise [4].

In this Letter we present what to our knowledge is the first finite-size scaling analysis of dynamic critical phenomena for a system in which the nonequilibrium behavior is due to an explicit time dependence of the Hamiltonian, and for which the nonequilibrium states are nonstationary in time. This is the kinetic Ising model in an oscillating field. We focus on the nature of the dynamic phase transition (DPT) in this model, which was proposed by several workers based on numerical observations [5–7]. Our results clarify the nature of the DPT for this specific model and also provide evidence for the relevance of universality and finite-size scaling concepts to dynamic phase transitions in nonstationary systems.

The relaxation of kinetic Ising models prepared with all spins aligned in a strong field, which is suddenly reversed, models the dynamics of metastable phase decay in a *static* field [8,9]. Such simulations have been used to study magnetization switching in anisotropic ferromagnets. The hysteretic response to an *oscillating* field has been studied by, among others, mean-field [10–12] and Monte Carlo (MC) [5–7,13–16] methods, and some of the results have been used to analyze hysteresis loops from experiments on Fe and Co ultrathin films [17,18]. The relevance of our results thus extends beyond nonequilibrium statistical mechanics to the numerous fields in which kinetic Ising systems are used to model specific systems.

A dynamic phase transition, in which the period-averaged magnetization Q passes from a disordered state ($Q = 0$) to an ordered state ($Q \neq 0$), has been observed in mean-field [10–12] and MC studies [5–7]. The location of this transition depends on temperature, field amplitude, and frequency. It can be intuitively understood as a competition between two time scales: the period of the external field, $2\pi/\omega$, and the average lifetime, $\langle \tau(H_0) \rangle$, of the metastable phase following instantaneous field reversal from H_0 to $-H_0$. If $2\pi/\omega < \langle \tau(H_0) \rangle$ the magnetization cannot fully switch signs within a single period, and $Q \neq 0$. If $2\pi/\omega > \langle \tau(H_0) \rangle$ the magnetization follows the field, and $Q = 0$.

Our previous work on Ising models in sudden field-reversal simulations has identified distinct decay regimes in which the decay of the metastable phase proceeds through nucleation and growth of one or more compact droplets of the stable phase. These different regimes are characterized in great detail in Refs. [8,9]. For this Letter, the most important result from past work is that the metastable decay mode may change *drastically* as the temperature, field strength, and/or system size are varied. At weak fields and/or small system size, the decay proceeds by the nucleation and growth of a *single* droplet of the stable phase [single-droplet (SD) regime]. For stronger fields and/or larger systems many droplets contribute to the metastable decay [multidroplet (MD) regime]. The MD decay mode can be accurately described by the classical "Avrami's law" for nucleation and growth [19]. For each of these regimes, the statistical properties of the lifetime of the metastable phase are different, leading to very different responses in both static and time-varying [13,14] fields.

Here we emphasize two main points. First, care must be exercised in determining the temperature and field dependence of the DPT boundary for a fixed frequency, because the response of the model can change qualitatively when the temperature and field strength are changed. In fact, our simulations show evidence of a DPT for the MD regime *only*. Second, if the DPT in the kinetic Ising model is a true critical phenomenon, then the theoretical machinery of finite-size scaling for equilibrium phase transitions should be applicable.

The model used is a kinetic, nearest-neighbor Ising ferromagnet on a square lattice with periodic boundary conditions and Hamiltonian $\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - H(t) \sum_i s_i$. Here $s_i = \pm 1$, $\sum_{\langle ij \rangle}$ runs over all nearest-neighbor pairs, and $\sum_i$ runs over all L^2 lattice sites. The ferromagnetic coupling is $J = 1$. Each spin is subject to an oscillating field $H(t) = -H_0 \sin(\omega t)$. We measure the time-dependent magnetization per site, $m(t) = (1/L^2) \sum_{i=1}^{L^2} s_i(t)$, using the Glauber single-spin-flip Monte Carlo dynamic [20] with updates at randomly chosen sites. Each attempted spin flip from s_i to $-s_i$ is accepted with probability $W(s_i \rightarrow -s_i) = \exp(-\beta \Delta E_i)/[1 + \exp(-\beta \Delta E_i)]$. Here ΔE_i is the energy change of the system resulting from an accepted spin flip, and $\beta = 1/k_B T$, where k_B is Boltzmann's constant. The time unit is one Monte Carlo step per spin (MCSS).

All simulations are performed for three system sizes: $L = 64$, 90, and 128 at $T = 0.8T_c$. This temperature is sufficiently far below T_c, the critical temperature of the Ising model in zero field, that the thermal correlation length is small compared to L and the critical droplet radius. We choose $H_0 = 0.3J$. This field amplitude is such that, for simulations at $0.8T_c$ in a static field of $H = H_0$, all three system sizes are in the MD regime. To obtain the raw time-series data, the system was initially prepared with either a random arrangement of up and down spins with $m(t = 0) \approx 0$ or a uniform arrangement with all spins up, $m(t = 0) = 1$. We recorded $m(t)$ for several values of ω for approximately 1.7×10^7 MCSS (for the lowest frequencies, 5.9×10^5 MCSS). Each of these raw data files store t, $H(t)$, and $m(t)$ in increments of 1 MCSS and contain thousands of field periods. The largest files are about 800 megabytes and required 9 days (one month) to run for $L = 64$ ($L = 128$) on a single node of an IBM sp2. This is one of the most extensive MC simulations of hysteresis in Ising systems with finite temperature to date.

It is useful to think of hysteresis as a competition between two time scales: $2\pi/\omega$ and $\langle \tau(H_0) \rangle$. Therefore we report all of our results in terms of the dimensionless period,

$$R = (2\pi/\omega)/\langle \tau(H_0) \rangle. \tag{1}$$

The average lifetime has been measured in field reversal simulations to be $\langle \tau(H_0) \rangle \approx 74.5$ MCSS and is independent of L in the MD regime.

The order parameter of the dynamic phase transition is the period-averaged magnetization,

$$Q = \frac{\omega}{2\pi} \oint m(t) \, dt. \tag{2}$$

This definition removes the field oscillations, so that Q is a stationary stochastic process. For each frequency we obtain the probability density of Q by constructing a histogram of the Q values calculated from each period in the corresponding time series.

Figure 1 shows the probability densities of Q, $P(Q)$, for all three system sizes at a frequency near the transition. (The details of locating the transition frequency are given below.) Correlation times that are significant portions of the total run length are manifest in the remaining asymmetry of the distributions. Estimating the correlation time from the asymmetry in $P(Q)$ we find it to be between 1% and 10% of the total simulation length near the transition. Away from the transition it decreases rapidly. The behavior is reminiscent of the critical slowing down seen in equilibrium simulations, and even our extensive simulations are not long enough to be fully ergodic. The asymmetry in the distributions is *not* sensitive to the initial condition of the time series.

At a second-order phase transition there is a divergence in the susceptibility. For equilibrium systems, the fluctuation-dissipation theorem relates the susceptibility to fluctuations in the order parameter. For the present system, it is not obvious what the field conjugate to Q might be. Therefore we cannot measure the susceptibility directly. However, we can calculate the variance in $|Q|$, $\mathrm{Var}(|Q|)$, as a function of frequency and study its system size dependence. We define X as

$$X = L^2 \, \mathrm{Var}(|Q|) = L^2 [\langle Q^2 \rangle - \langle |Q| \rangle^2]. \tag{3}$$

If the system were to obey a fluctuation-dissipation relation, X would be proportional to the susceptibility and both would scale with L in the same manner. Figure 2 shows X vs $1/R$ for three system sizes. For all three values of L, X displays a prominent peak which increases in height with increasing system size. This clearly shows finite-size effects in X and implies the existence of a divergent length associated with the order-parameter correlation function near the dynamic transition. The
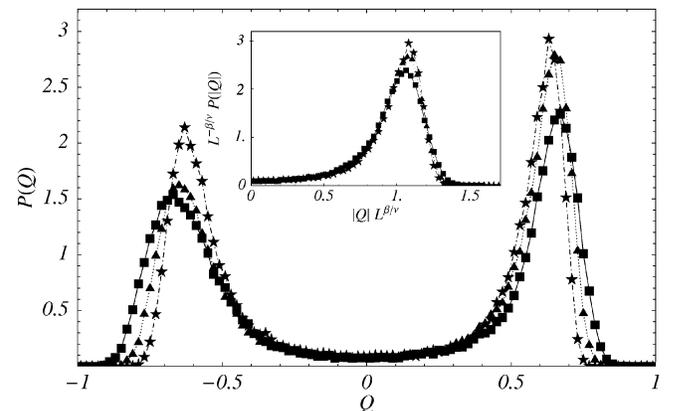

FIG. 1. Probability densities of the period-averaged magnetization, $Q = (\omega/2\pi) \oint m(t) \, dt$ for three system sizes, $L = 64$ (squares), 90 (triangles), and 128 (stars). The frequency of the field, $1/R = 0.2910$ is near the transition. The asymmetric distributions indicate correlation times on the order of the simulation time even for our extremely long runs of 1.7×10^7 MCSS. Inset: Scaling function $L^{-\beta/\nu} P(|Q|)$ vs $L^{\beta/\nu} |Q|$. The value of the scaling exponent used is $(\beta/\nu)_{n=2} \approx 0.111$.
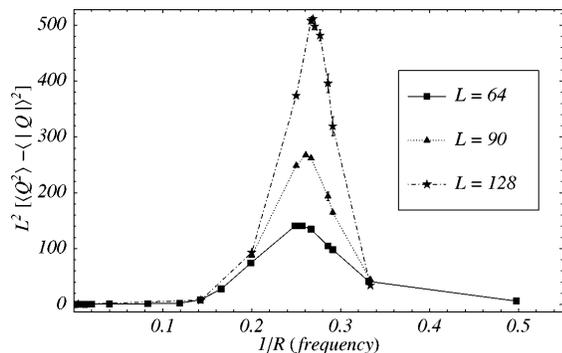
FIG. 2. $L^2 \operatorname{Var}|Q|$ vs dimensionless frequency, $1/R$. The "disordered phase" ($\langle|Q|\rangle = 0$) lies on the low-frequency side of the peaks. The "ordered phase" ($\langle|Q|\rangle \neq 0$) lies on the high-frequency side. Lines connecting data points are guides to the eye. The statistical error bars are estimated by partitioning the data into ten blocks. Error bars smaller than the symbol sizes are not shown.

observation that $P(|Q|)$ displays no peak near $|Q| = 0$ in the ordered phase is additional evidence for the second-order (as opposed to first-order) nature of this transition [21].

One would like to find the critical exponents associated with this transition, as well as the frequency at which the transition occurs. For the Ising model in zero field, T_c is exactly known. Then, one can use scaling relations which depend on T_c to directly calculate the critical exponents. In the present case, no exact solution exists for the transition frequency, and the scaling relations for X involve the peak heights and positions. Both of these are difficult to measure accurately, even from our extensive data. The cumulant intersection method [20,22] is useful for determining the location of a second-order transition when the critical exponents are not known. We define the "dynamic" fourth-order cumulant ratio as

$$U_L = 1 - \frac{\langle|Q|^4\rangle_L}{3\langle|Q|^2\rangle_L^2}, \qquad (4)$$

where $\langle|Q|^n\rangle = \int_0^\infty |Q|^n P(|Q|)\, d|Q|$. Figure 3 shows U_L vs $1/R$. Above the transition frequency, in the $\langle|Q|\rangle \neq 0$ phase, U_L approaches $2/3$. Below the transition frequency, in the $\langle Q\rangle = 0$ phase, U_L approaches 0. At the transition, the cumulant should have a nontrivial, fixed value, U^*. Therefore, the location of the cumulant intersection gives an estimate of the transition frequency without foreknowledge of the critical exponents. Because of the large spacing of our data and possible correction-to-scaling effects, we cannot identify a unique intersection point. We estimate the location of the intersection by the crossing of the two largest system sizes near $1/R_c \approx 0.2910$ with $U_L = U^* \approx 0.61$.

Having estimated the transition frequency, R_c^{-1}, we can approximate the critical exponents [20,22]. We obtain estimates for β/ν, γ/ν, and $1/\nu$ using the two largest system sizes. At the transition we use scaling relations for
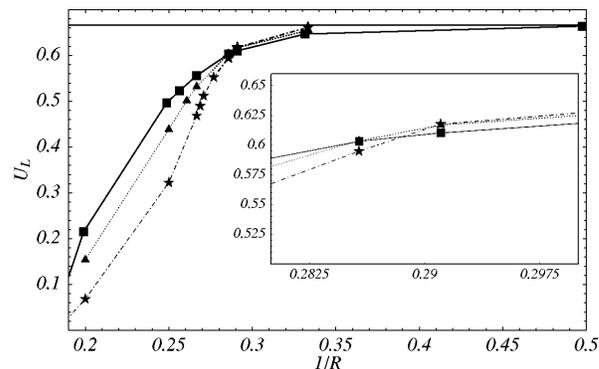

FIG. 3. Fourth-order cumulant ratio, U_L, vs dimensionless frequency, $1/R$, for $L = 64$, 90, and 128. We use the same system size symbols as in Fig. 1. The horizontal line marks $U_L = 2/3$. Lines connecting the data points are guides to the eye. Inset: Area close to the cumulant crossing.

the moments of the order parameter ($\langle|Q|^n\rangle \propto L^{-n(\beta/\nu)}$), the maximum value of the order-parameter fluctuations ($X_{\max} \propto L^{\gamma/\nu}$), and the position R_L^{-1} of the maximum value of the order parameter fluctuations for a particular system size ($|R_L^{-1} - R_c^{-1}| \propto L^{-1/\nu}$). Using the scaling relations for either the second or fourth moments of $|Q|$ we estimate $(\beta/\nu)_{n=2} \approx 0.111$ and $(\beta/\nu)_{n=4} \approx 0.113$. Our estimates for the other exponents are $\gamma/\nu \approx 1.84$ and $\nu \approx 1.1$. Simulations with larger system sizes would be computationally prohibitive, and smaller system sizes would no longer be in the MD regime.

The scaled probability distributions of $|Q|$ are given in the inset of Fig. 1 after symmetrizing and scaling to demonstrate data collapse. The symmetrizing is equivalent to calculating the distribution for $|Q|$. The scaling form is derived in a fashion analogous to equilibrium finite-size scaling analysis of order-parameter distributions [20,22]. At the transition, we assume that the mean of the order parameter scales with L and define the scaling variable $\tilde{Q} = L^{\beta/\nu}|Q|$. Hence, the scaled probability density for $|Q|$ is given by

$$\tilde{P}_L(\tilde{Q}) = L^{-\beta/\nu} P(|Q|). \qquad (5)$$

The peak positions scale fairly well, the peak heights less so. This could be due to the following reasons. The frequency might be sufficiently far from the transition so that single-parameter scaling is not adequate. There might be corrections to the finite-size scaling that are large for these relatively small system sizes. Also, the lack of scaling for the peak heights could be due to the asymmetry in $P(Q)$ near the transition. Much longer simulations on larger lattices would be needed to resolve this issue.

We have also carried out an extensive study of hysteresis in a kinetic Ising model in the SD regime, where we have found evidence of stochastic resonance but *no* sign of a dynamic phase transition [13]. In the introduction we emphasized that the crossover between the SD and MD

decay regimes depends on temperature, field strength, and system size. Therefore, the very existence of the dynamic transition, as well as the details of its critical behavior, may depend sensitively on all of these parameters.

In conclusion, we have performed a finite-size scaling study of a kinetic Ising model in a sinusoidal field in order to clarify the nature of the dynamic phase transition conjectured by several authors [5,7,14]. For this Letter, we emphasize that all simulations were performed in the MD regime. The behavior of the order-parameter fluctuation, X, suggests a divergent correlation length near the transition frequency. This behavior motivates the application of finite-size scaling techniques for second-order phase transitions, analogous to those used to describe the ferromagnetic/paramagnetic transition in the Ising model in zero field. We use the cumulant of the order-parameter distributions to estimate the value of the transition frequency, $1/R_c \approx 0.2910$. Using scaling relations for the moments and fluctuations of the order parameter we estimate the critical exponents to be $\beta/\nu \approx 0.11$, $\gamma/\nu \approx 1.84$, and $\nu \approx 1.1$. Our results are close to the two-dimensional Ising values for the analogous exponents. Given our level of precision, the values appropriate to two-dimensional random percolation are also not excluded. The result, $2(\beta/\nu) + (\gamma/\nu) \approx 2.06 \approx d$, gives a tantalizing indication that hyperscaling may be obeyed. Also, our value for the cumulant intersection, $U^* \approx 0.61$, agrees with an extremely precise transfer matrix calculation of $U^* = 0.610\,690\,1(5)$ for the two-dimensional Ising model [23]. To precisely calculate the critical frequency, critical exponents, and determine the universality class of the transition would require simulations for more frequencies, larger systems, and for immensely longer run times to improve the statistics. More detailed analysis of this problem could resolve specific questions about the DPT for the Ising model, such as the purported existence of a tricritical point in the dynamic phase diagram [6], in addition to elucidating general questions concerning the nature of nonequilibrium phase transitions in models with explicitly time-dependent Hamiltonians.

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