Spinor Bose Condensates in Optical Traps

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(Received 18 March 1998)

We show that in an optical trap the ground states of spin-1 bosons such as ^{23}Na, ^{39}K, and ^{87}Rb can be either ferromagnetic or "polar" states, depending on the scattering lengths in different angular momentum channels. The collective modes of these states have very different spin character and spatial distributions. While ordinary vortices are stable in the polar state, only those with unit circulation are stable in the ferromagnetic state. The ferromagnetic state also has coreless (or Skyrmion) vortices like those of ^{3}He-*A*. [S0031-9007(98)06714-3]

PACS numbers: 03.75.Fi, 05.30.Jp

Recently, the MIT group succeeded in trapping a ^{23}Na Bose condensate by purely optical means [1]. This experiment has opened up a new direction in the study of confined dilute atomic gases. In conventional magnetic traps, the spins of the alkali atoms are frozen. As a result, even though the alkali atoms carry spins, they behave like scalar particles. In contrast, the spin of the alkali atoms are essentially free in an optical trap. The spinor nature of alkali Bose condensate can therefore be manifested [2]. Because of the wide range of hyperfine spins of the alkali bosons and fermions, the optical trap has provided great opportunities to study dilute quantum gases of atoms with large spins.

The purpose of this paper is to point out the general properties of the spinor Bose condensates. As we shall see, they possess a whole host of quantum phenomena absent in scalar condensates. These include changes in ground state structures with interaction parameters, vector and quadrupolar spin wave modes, topological and energetic instability of doubly quantized singular vortices, and the existence of coreless (or Skyrmion) vortices. All of these results are simple consequences of the effective low energy Hamiltonian of the system. The derivation of this Hamiltonian (which applies to bosons and fermions with arbitrary spins) and the realization of its limitations are therefore crucial for our discussions.

For simplicity, we shall consider bosons with hyperfine spin $f = 1$. This includes alkalis with nuclear spin $I = 3/2$ such as ^{23}Na, ^{39}K, and ^{87}Rb. Alkali bosons with $f > 1$ such as ^{85}Rb (with $I = 5/2$) and ^{133}Cs (with $I = 7/2$) have even richer structures and will be discussed elsewhere. To illustrate the fully degenerate spinor nature, and as a first step, we shall consider only the case of zero magnetic field. To reach a good approximation of this limit, the Zeeman energy must be much smaller than the interaction energy at the center of the trap (which is essentially the chemical potential μ). The condition is therefore $\gamma B \ll \mu$, where γ is the gyromagnetic ratio and B is the magnetic field. For clouds with 10^6 atoms, with scattering length $a_{\rm sc} = 100$ Å and trap frequency 200 Hz, this condition gives $B \ll 10^{-3}$ G. This bound can be further increased by increasing the trap frequency. Since the current capability of magnetic shielding can reach 10^{-5} G, a good approximation of the zero field limit is attainable.

Optical trap potentials and their spin dependence.—We begin by examining the confining potential produced by a laser with a linearly polarized electric field $\mathcal{E}(t)\hat{\mathbf{x}}$ and frequency ω. The potential seen by an atom in hyperfine state $|1, m\rangle$ is $U_m = -\frac{1}{2}\alpha_m \overline{\mathcal{E}^2}$,

$$\alpha_m = \sum_{\ell} e^2 \frac{|X_{\ell m}|^2 \omega_{\ell m}}{\omega_{\ell m}^2 - \omega^2}, \qquad \sum_{\ell} |X_{\ell m}|^2 \omega_{\ell m} = \frac{\hbar}{2M} Z . \tag{1}$$

where $\overline{(\cdots)}$ denotes the time average, α_m is the polarizability of the state $|1, m\rangle$, $\hbar\omega_{\ell m} > 0$ is the excitation energy from the ground state $|1, m\rangle$ to excited states $|\ell\rangle$, $eX_{\ell m}$ is the dipole matrix element between $|1, m\rangle$ and $|\ell\rangle$, and Z is the atomic number. We have also written the dipole sum rule in Eq. (1). To one part in 10^2 to 10^3, the dipole sum rule is saturated by the nS to nP transitions [3], where nS is the ground state electronic configuration. The sum rule in Eq. (1) can therefore be approximated as $\sum_{\ell \epsilon P} |X_{\ell m}|^2 \omega_{\ell m} = \frac{\hbar}{2M} Z$. In addition, the excitation energies $\omega_{\ell m}$ for different nS to nP transitions differ from each other only by the fine splitting, which is typically 10^{-3} smaller than the energy difference ($\hbar\omega_{PS}$) between nS to nP levels before they fine split. Thus, for a far detuned laser, Eq. (1) can be approximated as

$$\alpha_m = -\frac{1}{\omega_{PS}^2 - \omega^2}\left(\sum_{\ell \epsilon P} |ex_{\ell m}|^2 \omega_{\ell m}\right) + O\left(\frac{\Delta_{\rm fine}}{\Omega}\right), \tag{2}$$

where $\Delta_{\rm fine}$ is the fine splitting and $\Omega \equiv \omega_{PS} - \omega$ is the detuning frequency. To the extent that the transitions $nS \rightarrow n'P$ ($n' > n$) and the ratio $\Delta_{\rm fine}/\Omega$ can be ignored, the sum in the first term in Eq. (2) is then a constant and α_m (hence U_m) is independent of m.

Effective low energy Hamiltonian.—Alkali atoms (bosons or fermions) have two hyperfine multiplets. The energy difference between the higher and the lower

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multiplet (denoted as f_{high} and f_{low}, respectively) is many orders of magnitude larger than the frequencies of typical traps. Since the interaction between two alkali atoms depends on their electron spins (singlet or triplet), the hyperfine states of the atoms can be changed after the scattering. However, when the system is very cold, two atoms in f_{low} will remain in the same multiplet after the scattering since there is not enough energy to promote either atom to f_{high}. In contrast, energy conservation does not limit the production of f_{low} states from the scattering of two f_{high} states. As a result, in an optical trap, all atoms in the ground state will be in the lower multiplet. The low energy dynamics of the system is therefore described by a pairwise interaction that is rotationally invariant in the hyperfine spin space and preserves the hyperfine spin of the individual atoms. The general form of this interaction is $\hat{V}(\mathbf{r}_1 - \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)\sum_{F=0}^{2f} g_F \mathcal{P}_F$, where $g_F = 4\pi\hbar^2 a_F/M$, M is the mass of the atom, $\mathcal{P}_F$ is the projection operator which projects the pair 1 and 2 into a total hyperfine spin F state, and a_F is the s-wave scattering length in the total spin F channel. For bosons (or fermions), symmetry implies that only even (or odd) F terms appear in $\hat{V}$.

For a system of $f = 1$ bosons, we have $V = g_2\mathcal{P}_2 + g_0\mathcal{P}_0$. Likewise, the relation $\mathbf{F}_1 \cdot \mathbf{F}_2 = \sum_{F=0}^{2f} \lambda_F \mathcal{P}_F$, $\lambda_F \equiv \frac{1}{2}[F(F+1) - 2f(f+1)]$ becomes $\mathbf{F}_1 \cdot \mathbf{F}_2 = \mathcal{P}_2 - 2\mathcal{P}_0$. The relation $\sum_F \mathcal{P}_{F=0}^{2f} = 1$ becomes $1 = \mathcal{P}_2 + \mathcal{P}_0$. We then have (dropping the δ function) $V = c_0 + c_2\mathbf{F}_1 \cdot \mathbf{F}_2$, where $c_0 = (g_0 + 2g_2)/3$ and $c_2 = (g_2 - g_0)/3$. The Hamiltonian in the second quantized form is then

$$\mathcal{H} = \int d\mathbf{r}\left(\frac{\hbar^2}{2M}\nabla\psi_a^+ \cdot \nabla\psi_a + U\psi_a^+\psi_a + \frac{c_0}{2}\psi_a^+\psi_{a'}^+\psi_{a'}\psi_a + \frac{c_2}{2}\psi_a^+\psi_{a'}^+\mathbf{F}_{ab}\cdot\mathbf{F}_{a'b'}\psi_{b'}\psi_b\right), \quad (3)$$

where $\psi_a(\mathbf{r})$ is the field annihilation operator for an atom in hyperfine state $|1, a\rangle$ at point $\mathbf{r}$, $(a = 1, 0, -1)$, and U is the trapping potential. It is straightforward to generalize Eq. (3) to higher spins. By noting that $(\mathbf{S}_1 \cdot \mathbf{S}_2)^n = \sum_{F=0}^{2f} \lambda_F^n \mathcal{P}_F$, we have for the general case $V = \sum_{n=0}^{f} \times c_n(\mathbf{F}_1 \cdot \mathbf{F}_2)^n$, where c_n is a linear combination of the g_F's. Alternatively, it is useful to express the projection operator $\mathcal{P}_F$ in the second quantized form as $\mathcal{P}_F = \sum_{a=-F}^{F} \hat{O}_{Fa}^+\hat{O}_{Fa}$, where $\hat{O}_{Fa} = \sum_{a_1,a_2}\langle Fa|f, a_1; f, a_2\rangle \times \hat{\psi}_{a_1}\hat{\psi}_{a_2}$, and $\langle Fm|f, a_1; f, a_2\rangle$ is the Clebsch-Gordon coefficient for forming a total spin F state from two spin-f particles. The pair potential is then $\hat{V} = \sum_{F=0}^{2f} \times g_F \sum_{a=-F}^{F} \hat{O}_{Fa}^+\hat{O}_{Fa}$. Only even and odd F terms appear in $\hat{V}$ in the case of bosons and fermions.

Ground state structure.—It is convenient to write the Bose condensate $\Psi_a(\mathbf{r}) \equiv \langle\hat{\psi}_a(\mathbf{r})\rangle$ as $\Psi_a(\mathbf{r}) = \sqrt{n(\mathbf{r})} \times \zeta_a(\mathbf{r})$, where $n(\mathbf{r})$ is the density and ζ_a is a normalized spinor $\zeta^+ \cdot \zeta = 1$. The ground state structure of $\Psi_a(\mathbf{r})$ is determined by minimizing the energy with a fixed particle number, i.e., $\delta K = 0$, $K \equiv \delta\langle H - \mu N\rangle$, where μ is the chemical potential,

$$K = \int d\mathbf{r}\left(\frac{\hbar^2}{2M}(\nabla\sqrt{n})^2 + \frac{\hbar^2}{2M}(\nabla\zeta)^2 n - [\mu - U(\mathbf{r})]n + \frac{n^2}{2}[c_0 + c_2\langle\mathbf{F}\rangle^2]\right), \quad (4)$$

and $\langle\mathbf{F}\rangle \equiv \zeta_a^*\mathbf{F}_{ab}\zeta_b$. It is obvious that all spinors related to each other by gauge transformation $e^{i\theta}$ and spin rotations $\mathcal{U}(\alpha, \beta, \tau) = e^{-iF_z\alpha}e^{-iF_y\beta}e^{-iF_z\tau}$ are degenerate, where (α, β, τ) are the Euler angles. There are two distinct cases:

(I) Polar state [4]: This state emerges when $c_2 > 0$ (i.e., $g_2 > g_0$). The energy is minimized by $\langle\mathbf{F}\rangle = 0$. The spinor ζ and the density n^0 in the ground state are given by

$$\zeta = e^{i\theta}\mathcal{U}\begin{pmatrix}0\\1\\0\end{pmatrix} = e^{i\theta}\begin{pmatrix}-\frac{1}{\sqrt{2}}e^{-i\alpha}\sin\beta\\ \cos\beta\\ \frac{1}{\sqrt{2}}e^{i\alpha}\sin\beta\end{pmatrix}, \quad n^0(\mathbf{r}) = \frac{1}{c_0}[\mu - U(\mathbf{r}) - W(\mathbf{r})], \quad (5)$$

where $W(\mathbf{r}) = \frac{\hbar^2}{2M}\frac{\nabla^2\sqrt{n^0}}{\sqrt{n^0}}$. Note that ζ is independent of the Euler angle τ. The symmetry group of polar state is therefore $\mathrm{U}(1) \times S^2$, where U(1) denotes the phase angle θ and S^2 is a surface of a unit sphere denoting all orientations (α, β) of the spin quantization axis.

(II) Ferromagnetic state: This state emerges when $c_2 < 0$, or $g_0 > g_2$. The energy is minimized by making $\langle\mathbf{F}\rangle^2 = 1$. The ground state spinor and density are

$$\zeta = e^{i\theta}\mathcal{U}\begin{pmatrix}1\\0\\0\end{pmatrix} = e^{i(\theta-\tau)}\begin{pmatrix}e^{-i\alpha}\cos^2\frac{\beta}{2}\\ \sqrt{2}\cos\frac{\beta}{2}\sin\frac{\beta}{2}\\ e^{i\alpha}\sin^2\frac{\beta}{2}\end{pmatrix}, \quad n^0(\mathbf{r}) = \frac{1}{g_2}[\mu - U(\mathbf{r}) - W(\mathbf{r})]. \quad (6)$$

The direction of the spin is $\langle\mathbf{F}\rangle = \cos\beta\hat{\mathbf{z}} + \sin\beta \times (\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{y}})$. The combination $(\theta - \gamma)$ in Eq. (6) clearly displays a "spin-gauge" symmetry [2], i.e., the equivalence between phase change and spin rotation. Because of this symmetry, the distinct configurations of ζ (including the gauge) are given by the full range of the Euler angles. The symmetry group is therefore SO(3). As we shall see, this difference in symmetry between the polar and the ferromagnetic state leads to a fundamental difference in their vortices.

According to the latest estimate of Burke, Bohn, and Greene [5], the scattering of ^{23}Na is $a_2 = (52 \pm 5)a_B$ and $a_0 = (46 \pm 5)a_B$; and those for ^{87}Rb are $a_2 = (107 \pm 4)a_B$ and $a_0 = (110 \pm 4)a_B$, where a_B is the Bohr radius.

Because of the overlapping error bars of a_2 and a_0 in each case, one cannot be sure about the nature of their ground states. However, if the inequalities suggested by current estimate ($a_2 > a_0$ for ^{23}Na and $a_0 > a_2$ for ^{87}Rb) are true, then the condensates of ^{23}Na and ^{87}Rb are polar state and ferromagnetic state, respectively.

Collective modes of trapped spinor Bose condensates.—The equation of motion in zero field is

$$i\hbar\partial_t\hat{\psi}_m = -\frac{\hbar^2}{2M}\nabla^2\hat{\psi}_m + [U(\mathbf{r}) - \mu]\hat{\psi}_m + c_0(\hat{\psi}_a^+\hat{\psi}_a)\hat{\psi}_m + c_2(\hat{\psi}_a^+\mathbf{F}_{ab}\hat{\psi}_b)\cdot(\mathbf{F}\hat{\psi})_m\,. \tag{7}$$

To study the elementary excitations, we write $\hat{\psi}_m = \Psi_m^0 + \hat{\phi}_m$ and linearize Eq. (7) about the ground state Ψ^0.

(I) Polar state: Without loss of generality, we take $\hat{\mathbf{z}}$ as the spin quantization axis and $\zeta^T = (0, 1, 0)$, with the subscript "T," denotes the transpose. Using the expression of n^0 in Eq. (5) and the fact that $\langle\mathbf{F}\rangle = 0$, Eq. (7) becomes

$$i\hbar\partial_t\begin{pmatrix}\hat{\phi}_0\\ -\hat{\phi}_0^+\end{pmatrix} = -\frac{\hbar^2}{2M}\nabla^2\begin{pmatrix}\hat{\phi}_0\\ \hat{\phi}_0^+\end{pmatrix} + W(\mathbf{r})\begin{pmatrix}\hat{\phi}_0\\ \hat{\phi}_0^+\end{pmatrix} + n^0c_0\begin{pmatrix}\hat{\phi}_0 + \hat{\phi}_0^+\\ \hat{\phi}_0 + \hat{\phi}_0^+\end{pmatrix}, \tag{8}$$

$$i\hbar\partial_t\begin{pmatrix}\hat{\phi}_1\\ -\hat{\phi}_{-1}^+\end{pmatrix} = -\frac{\hbar^2}{2M}\nabla^2\begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_{-1}^+\end{pmatrix} + W(\mathbf{r})\begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_{-1}^+\end{pmatrix} + n^0c_2\begin{pmatrix}\hat{\phi}_1 + \hat{\phi}_{-1}^+\\ \hat{\phi}_1 + \hat{\phi}_{-1}^+\end{pmatrix}. \tag{9}$$

To linear order in $\hat{\phi}_m$, the density and spin fluctuations ($\delta\hat{n}$ and $\delta\hat{M}_\pm$) are related to $\hat{\phi}_0$ and $\hat{\phi}_\pm$ as $\delta\hat{n}(\mathbf{r}) = \sqrt{n^0(\mathbf{r})}\,(\hat{\phi}_0 + \hat{\phi}_0^+)$, $\delta\hat{M}_+ \equiv \delta(\hat{M}_x + i\hat{M}_y) = \sqrt{n^0(\mathbf{r})}\times(\hat{\phi}_1 + \hat{\phi}_{-1}^+)$, and $\hat{M}_- = \hat{M}_+^+$. Denoting the frequencies of $\hat{\phi}_0$ and $\hat{\phi}_\pm$ as ω_0 and $\omega_\pm$, it is easy to see that they are all of the Bogoliubov form in the homogeneous case ($W = 0$), where $\hbar\omega_0 = \sqrt{\epsilon_\mathbf{k}(\epsilon_\mathbf{k} + 2c_0n^0)}$, $\omega_\pm = \sqrt{\epsilon_\mathbf{k}(\epsilon_\mathbf{k} + 2c_2n^0)}$, and $\epsilon_\mathbf{k} = \hbar^2k^2/(2M)$.

In a harmonic trap, Eq. (8) is identical to the equation of the collective mode of a scalar boson. Using the method of Stringari [6], it is straightforward to show that for a large cloud (where n^0 in Eq. (5) is well approximated by the Thomas-Fermi expression $n^0(\mathbf{r}) = [\mu - U(\mathbf{r})]/c_0$ by ignoring $W(\mathbf{r})$ [7]), Eqs. (8) and (9) can be written as

$$\partial_t^2\delta\hat{n} = \nabla(c_0n^0\nabla\delta n), \qquad \partial_t^2\delta\hat{M}_\pm = \nabla(c_2n^0\nabla\delta\hat{M}_\pm)\,. \tag{10}$$

Stringari has shown that the density mode in Eq. (10) has a universal spectrum (i.e., interaction independent) with power law (hence extended) wave functions [6]. Since the spin wave modes $\delta M_\pm$ obey exactly the same equation as δn, except that c_0 is replaced by c_2, the quantum numbers and the wave functions of the spin wave modes are identical to those of the density mode, and

$$\omega_\pm^2 = \frac{c_2}{c_0}\,\omega_0^2 = \frac{a_2 - a_0}{2a_2 + a_0}\,\omega_0^2\,. \tag{11}$$

(II) Ferromagnetic state: We shall take $\zeta^T = (1, 0, 0)$. Using Eq. (6) and the fact that $\langle\mathbf{F}\rangle = \hat{\mathbf{z}}$, Eq. (7) becomes

$$i\hbar\partial_t\begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_0\\ \hat{\phi}_{-1}\end{pmatrix} = -\frac{\hbar^2}{2M}\nabla^2\begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_0\\ \hat{\phi}_{-1}\end{pmatrix} + W(\mathbf{r})\begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_0\\ \hat{\phi}_{-1}\end{pmatrix} + n^0\begin{pmatrix}g_2(\hat{\phi}_1 + \hat{\phi}_1^+)\\ 0\\ 2|c_2|\hat{\phi}_{-1}\end{pmatrix}. \tag{12}$$

To linear order in $\hat{\phi}_a$, the density, spin, and "quadrupolar" spin fluctuations are given by $\delta\hat{n} = \sqrt{n^0}\,(\hat{\phi}_1 + \hat{\phi}_1^+)$, $\delta\hat{M}_- = \sqrt{n^0}\,\hat{\phi}_0^+$, and $\delta\hat{M}_-^2 = 2\sqrt{n^0}\,\hat{\phi}_{-1}^+$. The frequencies of these modes will be denoted as ω_1, ω_0, and ω_{-1}, respectively. In the homogeneous case ($W = 0$), the density mode has a Bogoliubov spectrum $\hbar\omega_0 = \sqrt{\epsilon_\mathbf{k}^2 + 2g_2n^0\epsilon_\mathbf{k}^2}$, $\epsilon_\mathbf{k} = (\hbar k)^2/(2M)$. The spin wave $\delta\hat{M}_-$ has a free particle spectrum $\omega_0 = \epsilon_\mathbf{k}$. The frequency of $\delta\hat{M}_-^2$ is free-particle-like with a gap, $\omega_0 = \epsilon_\mathbf{k} + 2|c_2|n^0$.

In a harmonic trap, the density mode $\delta\hat{n}$ assumes the Stringari form because $\hat{\phi}_1$ obeys the same equation as scalar Bosons. The spectra of both $\hat{\phi}_0$ and $\hat{\phi}_{-1}$ are given by Schrödinger equations with potentials $W(\mathbf{r})$ and $W(\mathbf{r}) + 2|c_2|n^0(\mathbf{r})$, respectively. Outside the cloud, the potentials for both $\hat{\phi}_0$ and $\hat{\phi}_{-1}$ reduce to the harmonic potential $U(\mathbf{r}) - \mu$. For a large cloud, n^0 is well approximated by the Thomas-Fermi expression $n^0 = [\mu - U]/g_2$, except near the surface of the cloud. Calculating W with n^0, one finds that it becomes more attractive near the surface. The low energy modes of $\delta\hat{M}_-$ are therefore confined near the surface. For the quadrupolar spin waves, $\delta\hat{M}_-^2$, the low energy modes are even more confined to the surface because of the additional potential $2|c_2|n^0(\mathbf{r})$, which is much more repulsive than W inside the cloud. The spectrum of the surface modes of $\delta\hat{M}_-$ and $\delta\hat{M}_-^2$ therefore mimic their homogeneous counterpart, with the energy of $\delta\hat{M}_-^2$ shifted up from that of $\delta\hat{M}_-$ by an amount $\sim|c_2|n^0 \sim (|c_2|/g_2)\mu = (|c_2|/g_2)(R/a_T)^2\hbar\omega_T$, where R is the size of the cloud and $a_T = \sqrt{\hbar/M\omega_T}$ is the trap length.

Intrinsic stability of singular vortices with circulation ($\ell > 1$) in the ferromagnetic state.—The fundamental difference between the vortices of the polar and the ferromagnetic states can be illustrated by their superfluid velocities, $\mathbf{v}_s \equiv \frac{\hbar}{M}\zeta^+\nabla\zeta$. From Eqs. (5) and (6), they are

$$(\mathbf{v}_s)_{\text{polar}} = \frac{\hbar}{M}\nabla\theta\,, \qquad (\mathbf{v}_s)_{\text{ferro}} = \frac{\hbar}{M}[\nabla(\theta - \tau) - \cos\beta\nabla\alpha]\,, \tag{13}$$

where we have assumed that the Euler angles (α, β, τ) and θ are spatially varying functions. Unlike the polar state, the superfluid velocity $\mathbf{v}_s$ of the ferromagnetic state depends on spin rotations. This leads to the following remarkable property of the Bose ferromagnets: If too much vortex energy is stored in one spin component, the system can get rid of it by spin rotation. To illustrate this phenomenon, consider the following family of spinor states $\{\Psi_a(t) = \sqrt{n^0}\,\zeta_a(t)\}$ parametrized by a parameter t between 0 and 1,

$$\zeta^T(t) = \left(e^{i2m\phi}\cos^2\left(\frac{\pi t}{2}\right), e^{im\phi}\sqrt{2}\sin\left(\frac{\pi t}{2}\right) \times \cos\left(\frac{\pi t}{2}\right), \sin^2\left(\frac{\pi t}{2}\right)\right), \quad (14)$$

where $m > 0$ is an integer, ϕ is the azimuthal angle, and n^0 is the equilibrium density for the vortex state $\zeta^T(t = 0) = (e^{2m\phi i}, 0, 0)$ with $\ell = 2m$ circulation. As t evolves from 0 to 1, this $2m\pi$ vortex evolves continuously to the vortex free state $\zeta^T(t = 1) = (0, 0, 1)$ with a spin texture $\langle\mathbf{F}\rangle = \cos(\pi t)\hat{\mathbf{z}} + \sin(\pi t)[\cos(m\phi)\hat{\mathbf{x}} + \sin(m\phi)\hat{\mathbf{y}}]$. This shows that $2m\phi$ vortices are topologically unstable. If they are ever stable, it must be due to the existence of energy barriers preventing its collapse. Such barriers, however, are nonexistent. From Eq. (4), it is easily shown that for $0 < t < 1$, $\frac{dK}{dt} = -\frac{\pi}{2}\frac{\hbar^2}{2M} \times \int d\mathbf{r}\, n^0(\mathbf{r})\, |\nabla 2m\phi|^2 (c + 2c^3)s < 0$, where $c \equiv \cos(\frac{\pi t}{2})$, $s^2 = 1 - c^2$. The fact that K decreases monotonically with increasing t shows the absence of energy barriers.

By multiplying Eq. (14) by $e^{i\phi}$, one also obtains a family connecting a vortex with $2m + 1$ circulation $\zeta^T(t = 0) = e^{i(2m+1)\phi}(1, 0, 0)$ to a vortex with unit circulation $\zeta^T(t = 1) = e^{i\phi}(1, 0, 0)$. It can be easily shown as in the $2m\phi$ case that $dK/dt < 0$ for $m > 0$. This means that all $2(2m + 1)\pi$ vortices will collapse into a 2π vortex. The 2π vortex, however, can not be deformed continuously into a uniform state [8].

Coreless (or Skyrmion) vortices.—Equation (13) shows that spin variations in the ferromagnetic states in general lead to superflows. To illustrate further, consider the condensate $\zeta(\mathbf{r})^T = (\cos^2\frac{\beta}{2}, \sqrt{2}\,e^{i\phi}\sin\frac{\beta}{2}\cos\frac{\beta}{2}, e^{2i\phi}\sin^2\frac{\beta}{2})$, where $\beta = \beta(r)$ is an increasing function of r starting from $\beta = 0$ at $r = 0$. The spin texture and superfluid velocity of this condensate are both cylindrically symmetric, $\langle\mathbf{F}\rangle = \hat{\mathbf{z}}\cos\beta + \sin\beta(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}})$, and $\mathbf{v}_s = \frac{\hbar}{Mr}(1 - \cos\beta)\hat{\phi}$. When $\beta(r)$ reaches $\pi/2$, $\mathbf{v}_s$ becomes the velocity field of a singular vortex. However, the singularity of the usual 2π vortex is absent because $\mathbf{v}_s$ vanishes instead of diverges at $r = 0$. This phenomena is identical to that of superfluid ^{3}He-A, which has exactly the same angular momentum texture, superfluid velocity, and topological instability [9,10]. In the case of ^{3}He-A, it is known that external rotations can distort the texture so as to generate a velocity field $\mathbf{v}_s$ to mimic the external rotation as closely as possible [9]. Such textural distortions will occur here for the same energetic reasons.

In current experiments, the condensates are first produced in a magnetic trap and then loaded into an optical trap. Because of the way they are produced, the condensates will in general carry a net magnetization. If the spin relaxation time is sufficiently long, the condensate will behave like a ferromagnetic state even its ground state is a polar state. In this case, the vortex phenomena discussed above will apply.

I thank Wolfgang Ketterle, Dan Stamper-Kurn, Eric Cornell, Carl Wieman, and Greg Lafayatis for valuable discussions. Special thanks are due to John Bohn, Jim Burke, and Chris Greene for kindly providing me with their estimates of scattering lengths. Part of this work was done during a visit to JILA in December 1997. I thank Eric Cornell and Carl Wieman for their hospitality. This work is supported by the NASA Grant No. NGA8-1441 and partly by NSF Grant No. DMR-9705295.

Notes added.—At the time of submission of this paper, a preprint by T. Ohmi and K. Machida appeared (cond-mat/9803160). These authors have studied the same model in finite field and in the absence of a trap. Our zero field results of homogeneous systems agree.

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