Entropic Lower Bound for the Quantum Scattering of Spinless Particles

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In this paper the *angle-angular momentum entropic lower bound* is proved by using Tsallis-like entropies and Riesz theorem for the quantum scattering of the spinless particles. Numerical estimations of the scattering entropies, as well as an experimental test of the *state-independent* entropic lower bound, are obtained by using the amplitude reconstruction from the available phase shift analyses for the pion-nucleus scatterings. A standard interpretation of these results in terms of the optimal state dominance is presented. Then, it is shown that experimental pion-nucleus entropies are well described by optimal entropies and that the experimental data are consistent with the *principle of minimum distance in the space of scattering states.* [S0031-9007(98)08033-8]

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Over the past two decades there has been increasing interest [1] in the investigation of the *entropic uncertainty relations* introduced in Ref. [2]. In this paper the entropic lower bound for the quantum scattering [3] is investigated in a more general form by introducing *Tsallis-like entropies* (see Refs. [4,5]). Hence, by using *Riesz theorem* [6], the *state-independent angle-angular momentum entropic lower bound* is proved for the scattering of spinless particles. The results on the first experimental test of this *state independent entropic bound* in pion-nucleus scattering are obtained by calculations of the scattering entropies from the experimental available *phase shifts* [7–11]. Moreover, comparisons of these results with the optimal state [12] prediction are presented.

Angular entropy S_θ.—The *informational angular entropy S_θ* of any quantum scattering states is defined as in Ref. [3] by

$$S_\theta = -\int_{-1}^{1} dx\, P(x) \ln P(x)\,, \tag{1}$$

where $P(x)$ is the angular distribution defined in terms of the differential cross section by

$$P(x) = \frac{2\pi}{\sigma_{\rm el}} \frac{d\sigma}{d\Omega}(x), \qquad \int_{-1}^{1} P(x)\, dx = 1\,. \tag{2}$$

More general, we can define the *Tsallis-like angular entropies* [4,5]

$$S_\theta(q) = \frac{1}{q-1}\left\{1 - \int_{-1}^{1} dx[P(x)]^q\right\}, \qquad q \in R \tag{3}$$

with the property

$$\lim_{q\to 1} S_\theta(q) = S_\theta(1) = S_\theta\,. \tag{4}$$

The angular momentum entropy S_L.—Now, let us consider the case when the scattering amplitude $f(x)$ of the spinless particles is developed in partial amplitudes as follows:

$$f(x) = \sum_{l=0}^{L} (2l+1) f_l P_l(x), \qquad x \in [-1,1],\ f_l \in C\,, \tag{5}$$

where $L+1$ is the number of partial amplitudes f_l, $P_l(x)$, $l = 0, 1, \ldots, L$, are Legendre polynomials. Then, the *Fourier coefficients* or the *partial amplitudes f_l* are expressed as

$$f_l = \frac{1}{2}\int_{-1}^{+1} f(x) P_l(x)\, dx, \qquad f_l \in C\,. \tag{6}$$

Hence, we define the *angular momentum entropy S_L* by

$$S_L = -\sum_{l=0}^{L} (2l+1)\cdot p_l \cdot \ln p_l\,, \tag{7}$$

where the partial probabilities p_l are defined by

$$p_l = 4\pi \frac{|f_l|^2}{\sigma_{\rm el}}, \qquad \sum_{l=0}^{L} (2l+1) p_l = 1\,. \tag{8}$$

Of course, in this case, the *Tsallis-like angular momentum entropies* for the scattering process can be defined as follows:

$$S_L(q) = \frac{1}{q-1}\left\{1 - \sum_{l=0}^{L} (2l+1)[p_l]^q\right\}, \qquad q \in R \tag{9}$$

with the property

$$\lim_{q\to 1} S_L(q) = S_L(1) = S_L\,. \tag{10}$$

The angle-angular momentum entropy $S_{\theta L}$.—The entropies (1) and (7) are defined as natural measures of the uncertainties corresponding to the distributions of probabilities $P(x)$ and p_l, respectively. If we are interested to obtain a measure of uncertainty of the simultaneous realization of the probability distributions $P(x)$ and p_l, then we must calculate the entropy corresponding to the product of these distributions: $P(x,l) = P(x)\cdot p_l$. It is easy to verify that the *angle-angular momentum entropy* is given by

$$S_{\theta L} = -\sum_{l=0}^{L} (2l+1) \int_{-1}^{1} dx\, P(x,l) \ln[P(x,l)] = S_\theta + S_L\,. \tag{11}$$

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In this case the Tsallis-like entropies for the scattering of spinless particles is given by

$$S_{\theta L}(q) = \frac{1}{q-1}\left\{1 - \sum_{l=0}^{L} p_l^q \int_{-1}^{1} dx[P(x)]^q\right\} = S_\theta(q) + S_L(q) + (q-1)S_\theta(q)S_L(q), \quad q \in R \tag{12}$$

with the property

$$\lim_{q\to 1} S_{\theta L}(q) = S_{\theta L}(1) = S_{\theta L} = S_\theta + S_L. \tag{13}$$

Therefore, the index $q \neq 1$ controls the degree of entropy nonextensivity reflected in the pseudoadditivity entropy rule (12).

Entropic inequalities.—It is interesting to present here the following generalized entropic inequalities for the Tsallis-like entropies.

$$\frac{1}{q-1}[1 - K^{q-1}(1,1)] \le S_\theta(q) \le \frac{1}{q-1}[1 - 2^{1-q}], \tag{14}$$

for $q > 0$, and

$$S_L(q) \le \frac{1}{q-1}\{1 - [2K(1,1)]^{1-q}\}, \tag{15}$$

$q > 0$.

The proof of the lower bound (14) is provided by considering the condition that $P(x)$ has, everywhere, a finite magnitude, i.e.,

$$P(x) \le P(1) = K(1,1) = \frac{1}{2}(L_0 + 1)^2 = \frac{2\pi}{\sigma_{\rm el}}\frac{d\sigma}{d\Omega}(1). \tag{16}$$

Next, the upper bounds (14) and (15) are optimal bounds which can be obtained via Lagrange multipliers by extremizing the Tsallis-like entropies subject to the normalization constraints (2) and (8), respectively.

State independent angle-angular momentum entropic lower bound.—Here we present the following entropic inequalities, namely, *state-independent entropic lower bound*

$$\ln 2 \le S_\theta + S_L. \tag{17}$$

A general proof of (7) can be obtained by applying the following *Riesz theorem* (see Theorem 2.8 from Ref. [6], p. 102).

Indeed, by using the relations

$$\left[\int P^m(x)\,dx\right]^{1/2m} = [1 + (1-m)S_\theta(m)]^{1/2m},$$

$$\left[\sum(2l+1)p_l^m\right]^{1/2m} = [1 + (1-m)S_L(m)]^{1/2m}, \quad m = p, q, \tag{18}$$

from the *Riesz Theorem 2.8* (Ref. [6], p. 102) (with $p \to 2p$ and $p' \to 2q$, so that $p^{-1} + q^{-1} = 2$) we get the following general result.

(p,q)*-Entropic bound: (i) Let* $f \in L^p(-1,+1)$, $\frac{1}{2} < p \le 1$ *be the scattering amplitude satisfying (5) with the Fourier coefficients given by (6). If the scattering Tsallis-like entropies are defined by Eqs. (3) and (9), respectively, then the following entropic inequalities hold:*

$$[1 + (1-q)S_L(q)]^{1/2q} \le \exp\left\{\left[\frac{p-1}{2p}\right]\ln 2\right\} \times [1 + (1-p)S_\theta(p)]^{1/2p} \tag{19}$$

for any q defined by the relation $\frac{1}{2p} + \frac{1}{2q} = 1$.

(ii) *For any finite sequence* f_l *with finite* $[1 + (1-p)S_L(p)]^{1/2p}$ *there is an* $f \in L^q(-1,+1)$ *satisfying* (5) *for which*

$$[1 + (1-q)S_\theta(q)]^{1/2q} \le \exp\left\{\left[\frac{p-1}{2p}\right]\ln 2\right\} \times [1 + (1-p)S_L(p)]^{1/2p} \tag{20}$$

where $\frac{1}{2p} + \frac{1}{2q} = 1$.

Hence, in the limit $p \to 1$ and $q \to 1$, from (19) [or from (20)], by developing in powers of Δp ($\Delta q = -q^2\Delta p/p^2$) and considering only the first terms, we get the lower bound (17).

Experimental tests.—The results presented in this paper are valid for the strong hadron-hadron, hadron-nucleus, or nucleus-nucleus scatterings for the spinless hadrons and only when the electromagnetic scattering contributions are subtracted from the experimental data. Hence, in the case of the hadron-nucleus (with atomic number Z and mass number A) scattering and nucleus-nucleus scattering a Z dependence of the experimental entropies (1) and (7) are expected to be observed only as a consequence of a violation of the charge independence of the nuclear forces while the A dependence of these entropies can be observed explicitly or is implicitly included via the optimal cutoff parameter L_0 [see Eq. (16)]. Here, for numerical investigation of our results it is interesting to calculate the entropies (1) and (7) by reconstruction of the hadron-nucleus scattering amplitudes using the available experimental phase shifts [7–10] for the π^0-^{4}He, π^0-^{12}C and π^0-^{16}O, π^0-^{40}Ca scatterings. The results obtained in this way are presented in Figs. 1 and 2 as functions of the *optimal angular momentum* L_0 which is obtained from the same phase shifts [7–11] by formula [see Eq. (16)]

$$L_0 = \left\{\frac{4\pi}{\sigma_{\rm el}}\frac{d\sigma}{d\Omega}(1)\right\}^{1/2} - 1. \tag{21}$$

Therefore, in Fig. 1 we presented the results of the first experimental test of the *state independent entropic lower bound (17)* in the pion-nucleus scattering. From Fig. 1 we see that this *entropic lower bound* is clearly experimentally verified with high accuracy. From Fig. 2 we see that the experimental scattering entropies (S_θ, S_L) for the π^0-^{4}He, π^0-^{12}C and π^0-^{16}O, π^0-^{40}Ca scatterings are well

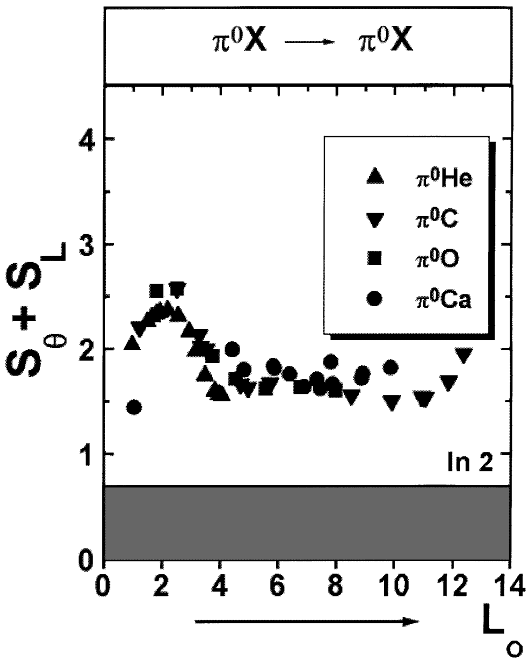

FIG. 1. The experimental tests of the *state independent entropic lower bound:* $S_\theta + S_J \geq \ln 2$, for the π^0-^{4}He, π^0-^{12}C and π^0-^{16}O, π^0-^{40}Ca scatterings, calculated by using Eqs. (1) and (7) and the experimental pion-nucleus phase shifts from Refs. [7–11], as functions of L_0.

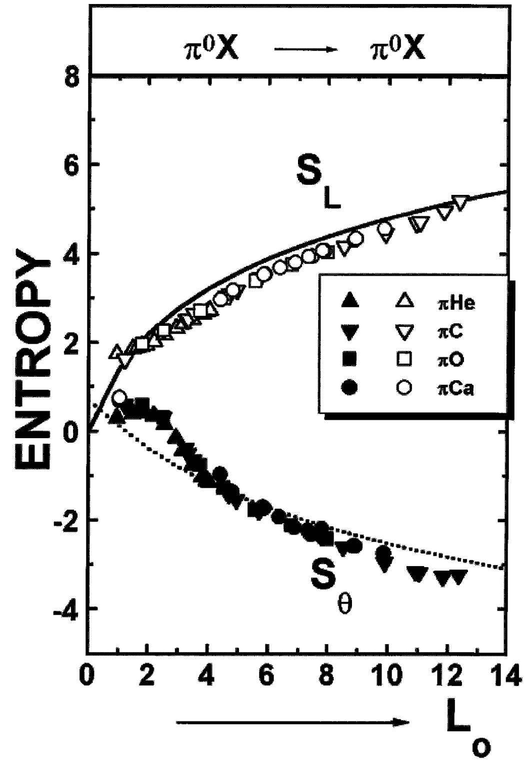

FIG. 2. The experimental entropies S_θ and S_L, obtained by using Eqs. (1) and (7) and the experimental phase shifts from Refs. [7–11] for the π^0-^{4}He, π^0-^{12}C and π^0-^{16}O, π^0-^{40}Ca scatterings, are compared with the theoretical optimal state predictions (22) (dotted curve) and (23) (full curve).

described (the full and dotted curves) by the following *optimal entropies* (22) and (23):

$$S_\theta^{o1} = -\int_{-1}^{1} dx\, \frac{[K(x,1]^2}{K(1,1)} \ln\left[\frac{[K(x,1]^2}{K(1,1)}\right], \qquad (22)$$

$$S_L^{o1} = \ln\{[L_0 + 1]^2\}. \qquad (23)$$

These entropies correspond to the optimal scattering state (26) from Ref. [12] where the *reproducing kernel functions* $K(x,y)$ are given in Eqs. (22) and (23) of Ref. [12]. The values of optimal $(S_\theta^{o1}, S_L^{o1})$, entropies for the scattering of spinless particles, are obtained by numerical integration and direct, respectively, and are presented in Table I only for $0 \leq L_0 \leq 12$. Clearly, the fact that the experimental entropies do not depend significantly on the atomic number A is a direct consequence of the *optimal state dominance* since in this case the entropies of all hadron nuclei as a function of the variable L_0 must be concentrated around the optimal values (22),(23) given in Table I.

Now, in order to see why the experimental entropies are well described by the optimal entropies $(S_\theta^{o1}, S_L^{o1})$, entropies (22) and (23), we observe that the entropy S_L (7) is similar to the Boltzmann entropy with a maximum value given by the logarithm of *number of the optimal states.* Indeed, from (15) we have

$$S_L \leq S_L^{o1} = \ln[2K(1,1)], \qquad (24)$$

where $2K(1,1) = \sum(2l + 1) = (L_0 + 1)^2$ is the *number of optimal scattering states* participating at the scattering process. This result allows us to conclude that the *optimal state* (26) from Ref. [12] is the state of *equilibrium* of the angular momenta channels considered as a *quantum statistical ensemble.* Hence, the *optimal angular distribution* $P^{o1}(x) = [K(x,1)]^2/K(1,1)$ can be considered as a signature of this *equilibrium distribution* of the L channels.

From Fig. 2, we see that the experimental values of (S_θ, S_L) *entropies* for the pion-nucleus scatterings are systematically described by the *optimal entropies* $(S_\theta^{o1}, S_\theta^{o1})$ at all available pion kinetic energies. In this sense the results obtained here can also be considered as new experimental signatures for the validity of the *principle of minimum distance in space of scattering states* even in the crude form [12].

The extension of the *optimal state* analysis to the *generalized nonextensive statistics case* ($q \neq 1$) [13], as well as a test of the entropic inequalities (19),(20) for $q \neq 1$, can be obtained in a similar way by using the following *nonextensive optimal entropies:* $S_\theta^{o1}(q) = \frac{1}{q-1}\{1 - \int_{-1}^{+1} dx[(K(x,1))^2/K(1,1)]^q\}$ and $S_L^{o1}(q) = \frac{1}{q-1}\{1 - [2K(1,1)]^{1-q}\}$. Since this kind of analysis is more involved the numerical examples for $q \neq 1$ will be given in a more extended paper.

Finally, we believe that the results obtained here are encouraging for further investigations of the entropic

TABLE I. The optimal entropies S_L^{o1}, S_θ^{o1}, $S_L^{o1} + S_\theta^{o1}$, corresponding to different optimal angular momentum L_0, calculated by using Eqs. (22), (23), and Ref. [12].

L_0	S_θ^{o1}	S_L^{o1}	$S_L^{o1} + S_\theta^{o1}$	L_0	S_θ^{o1}	S_L^{o1}	$S_L^{o1} + S_\theta^{o1}$
0	0.693	0	0.693	13	−2.970	5.278	2.308
1	0.128	1.386	1.514	14	−3.098	5.416	2.318
2	−0.385	2.197	1.812	15	−3.219	5.545	2.326
3	−0.806	2.773	1.966	16	−3.334	5.666	2.333
4	−1.158	3.219	2.061	17	−3.442	5.781	2.339
5	−1.460	3.584	2.124	18	−3.544	5.889	2.345
6	−1.722	3.892	2.170	19	−3.641	5.992	2.351
7	−1.955	4.159	2.204	20	−3.734	6.089	2.355
8	−2.164	4.394	2.231	21	−3.823	6.182	2.360
9	−2.353	4.605	2.253	22	−3.908	6.271	2.363
10	−2.526	4.796	2.270	23	−3.989	6.356	2.367
11	−2.685	4.970	2.285	24	−4.068	6.438	2.370
12	−2.832	5.130	2.298	25	−4.143	6.516	2.373

uncertainty relations as well as of the principle of minimum distance in space of states not only in the elementary particle physics but also in other domains of science such as in genetics, biology (see, e.g., Ref. [14]), etc.

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