Finsler Geometric Local Indicator of Chaos for Single Orbits in the Hénon-Heiles Hamiltonian

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Translating the dynamics of the Hénon-Heiles Hamiltonian as a geodesic flow on a *Finsler manifold*, we obtain a *local and synthetic* geometric indicator of chaos (GIC), which represents a link between local quantities and asymptotic behavior of orbits and gives a strikingly evident, one-to-one, correspondence between geometry and instability. Going beyond the results attained using the customary dynamical approach and improving also on the global criteria established within the Riemannian framework, the GIC is able to discriminate between regular and stochastic orbits on a given energy surface, simply on the basis of the value it assumes along a relatively small piece of the trajectory, without long integrations of the dynamics and without any reference to a perturbation. [S0031-9007(98)07882-X]

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The presence of instability in dynamical systems (DS's) is usually recognized *a posteriori*, looking at the long time evolution of small perturbations, whose average exponential growth is interpreted as *the* signature of chaos. Skipping here most of the issues related to the universal meaning attributed to it, we note how the procedures and tools that have at the grounds such a criterion alone manifest their limitations whenever the behavior of DS's at the boundary between quasiintegrability and stochasticity is investigated (e.g., [1,2]). Instead, the search for *a priori, synthetic* signatures of chaos dates back to Toda [3] and continues, across interesting investigations on the mechanisms of transition to stochasticity (see, e.g., [4]), up to the recently revived Riemannian geometrodynamical approach (GDA) [5,6]. Although most of its results relate to high dimensional Hamiltonian systems (for which some approximations are justified by the large number of d.o.f. or some weak form of the ergodic hypothesis), more recently this approach has been tested also for small DS's, giving outcomes clearly supporting its reliability [2,7]. Nevertheless, if in the case of large DS's considered the agreement between the GDA and the customary tools used to detect chaos has been revealed to be rather satisfactory, at least as long as the approximations are well justified [8], some discrepancies emerge in the case of few d.o.f. systems [2,9]. For the latter, the GDA provided an alternative way to recover most of the results obtained with the tangent dynamics equations, suggesting deeper hints for the understanding of the sources of chaos and giving in addition some global criteria to single out a transition in the overall behavior [7]. However, this criterion is unable to correctly detect the occurrence of chaos in single orbits [10], as it renounces, in principle, to intrinsically describe the behavior of individual trajectories. Within the Riemannian approach to few d.o.f. systems, this issue has been addressed, up to now, resorting only to a numerical procedure analogous to the integration of the tangent dynamics equations, whose results have been generally confirmed (though not always). Recently Kandrup [11] investigated in detail the relationships existing between *local* dynamical behavior and *local* geometric features of the Jacobi (Riemannian) manifold for some two-dimensional DS's, obtaining qualitative correlations among average curvature and its fluctuations and somewhat more ambiguous ones between curvature fluctuations and short time Lyapunov exponents. In summary, the Riemannian GDA has been able, up to now, either to intrinsically describe the average behavior of a DS or to single out the individual orbits instability *a posteriori*, as in the Hamiltonian description. Both of these approaches have remained unable to find an intrinsic and individual indicator of long-term behavior of orbits as the geometric indicator of chaos (GIC) here presented, built within the Finsler GDA and which cannot even be defined [for the Hénon-Heiles (HH) system] within the other previously mentioned frameworks. Notwithstanding its *local* character (in both spatial and temporal meanings), it is unambiguously related to Lyapunov characteristic numbers (LCN's), i.e., to asymptotic quantities, usually computed with reference to a perturbation. We claim then that it represents a strong indication (if not a proof) that the GDA is able not only to reproduce and to explain the results obtained with the usual tools, but even to go beyond them.

One of the main recent results of the GDA is the confirmation that the onset of unpredictability in the geometric transcription of realistic (large) DS's is driven by the mechanism of parametric instability [5,6] and thus differs completely from what occurs in the geodesic flows of abstract ergodic theory. Indeed, most of the phase space of large physical DS's is not characterized by (constant) negative curvatures, but stochasticity is caused by the quasirandom fluctuations of (mostly) positive curvatures. For few dimensional DS's, however, such random character cannot be assumed and instead is parametric resonance, similar to that occurring in the Mathieu equation, to bring about instability [6,7,12].

Nevertheless, we find below that even for 2 d.o.f. systems such a mechanism cannot be singled out in a naïve way and very intricate combinations of geometric features of the manifold are linked to instability in a nontrivial way.

We already discussed the motivations for an extension of the GDA to include non-Riemannian manifolds [12], and we also pointed out its somewhat greater effectiveness with respect to the usual tools in the computations of instability exponents [2,13]. We refer to [5,6,8] for a detailed description of the GDA, to [2,12,14] for its implementation within Finsler manifolds, and to [9] for a thorough discussion of most of the points here only sketched. Within the GDA the trajectories of a N d.o.f. system become the geodesics of suitable differential manifolds, which, in Finsler geometry, are $(N+1)$-dimensional and represent a generalization of Riemannian ones. The stability of the flow is determined by the Jacobi-Levi-Civita (JLC) equations of geodesic spread:

$$\frac{\nabla}{ds}\left(\frac{\nabla z^a}{ds}\right) + \mathcal{H}^a_c z^c = 0, \qquad (a = 0, 1, \ldots, N), \quad (1)$$

where z^a is the perturbation, ∇/ds is the covariant derivative along the geodesic, and the stability tensor $\mathcal{H}$ [6,8] derives from the (generalized) curvature tensor of the manifold. The Finsler time parameter $s \equiv s_F$ is defined through the Lagrangian function $\mathcal{L}$, as $ds_F = \mathcal{L}\,dt$ and possesses a built-in invariance with respect to an arbitrary rescaling of the Newtonian time t [15]. The *local* behavior of geodesics is determined by the eigenvalues of $\mathcal{H}$, which are the principal sectional curvatures (psc's) defined *by the given geodesic* on the manifold [8]. For a N degrees of freedom DS, once a geodesic is chosen, the Finsler stability tensor possesses $(N+1)$ eigenvalues $(\{\lambda_A\}, A = 0, 1, \ldots, N)$, one of which vanishes identically, $\lambda_0 \equiv 0$, associated with a neutral eigenvector, along the tangent to the geodesic. In the case of a standard Hamiltonian system (i.e., without gyroscopic terms), $H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} a_{ij} p^i p^j + \mathcal{U}(\mathbf{q})$ we have then [2,12]

$$\lambda_i = t'(B + t'\mu_i), \qquad (i = 1, \ldots, N), \qquad (2)$$

where B is related to the time derivatives of the Lagrangian, the $\{\mu_i\}$ are the eigenvalues of the Hessian $\mathcal{U}_{,ij}(\mathbf{q})$, and the prime denotes the derivative with respect to s_F. In this Letter we will deal with the well known two dimensional HH system, whose Hamiltonian is

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(x^2 + y^2 + p_x^2 + p_y^2\right) + x^2 y - \frac{y^3}{3}. \quad (3)$$

We find that for this DS (as well as for generic realistic ones, in spite of some persistent claims, e.g., [16]) negative curvatures are quite unable to explain the asymptotic character of orbits. The mechanism of parametric instability, due to fluctuations of usually positive curvatures [5,6] manifests *naïvely,* however, only for some, many d.o.f. systems being instead hardly perceived in this case. For example, the analysis of spectra [9] shows how intermingled and far from trivial are the relationships with the elementary theory of Mathieu-like equations. Such seemingly discouraging inconsistencies have driven us to check for different signatures of instability. The Finslerian approach allows us to consider, even for 2 d.o.f. Hamiltonians, the possible anisotropy of the manifold, which plays a crucial role, via the Schur theorem [8] in the mechanism of instability: fluctuating curvatures require also that the manifold is anisotropic. The connection between curvature variations along a geodesic and anisotropy, on one side, and growth and rotation of perturbation, on the other, is far from being clear and is currently investigated: how the former can interact to steer the latter can at the moment only be guessed. For a 2 d.o.f. system, the associated Finsler manifold is three dimensional and its curvature properties along a *given geodesic* are described by the two nonvanishing psc's, $\lambda_{1,2}$, which are invariant functions on the tangent space, representing the sectional (i.e., Gaussian) curvatures in the two planes defined from the tangent to the flow $\mathbf{u}$ and the two (nontrivial) eigenvectors of $\mathcal{H}$. Given them, we can characterize the way the geodesic explores the Finsler manifold through the (half) Ricci curvature along the flow $[\mathrm{Ric_F}(\mathbf{u})]$ and the anisotropy, $\kappa[\mathbf{q}(s), \mathbf{p}(s)]$ and $\vartheta[\mathbf{q}(s), \mathbf{p}(s)]$, respectively,

$$\kappa \doteq \frac{\lambda_1 + \lambda_2}{2} \equiv \frac{\mathrm{Tr}\,(\mathcal{H})}{2} = \frac{\mathrm{Ric_F}(\mathbf{u})}{2}; \qquad \vartheta \doteq \frac{\lambda_1 - \lambda_2}{2}. \quad (4)$$

An exhaustive statistical analysis of the behavior of κ and ϑ (or equivalently of $\lambda_{1,2}$) and the details of the logical path leading to the synthetic indicator are presented elsewhere [9]. We found that along a generic geodesic the $\{\lambda_i\}$ oscillate around their average values, the fluctuations of Ricci curvature in general turns out to be, however, much smaller than those of the sectional ones, which are, indeed, almost anticorrelated. Such an effect is particularly evident in the HH case, as $\Delta\mathcal{U} \equiv 2$. So, the manifold appears to be everywhere anisotropic but with psc's always (λ_1) or mostly (λ_2) positive. Fluctuations (and then anisotropy) increase with energy, showing a global qualitative change in correspondence of appearance of stochasticity [9]. Although smaller, the overall fluctuations of Ricci curvature seem nevertheless to influence appreciably the stability of geodesics. Moreover, Schur theorem asserts that the two quantities must be related. We then look for a relative measure of anisotropy fluctuations compared to overall curvature variations. Correlations between two quantities $A(s)$ and $B(s)$ reflect in the functional

$$\tilde{C}_S[A, B] \doteq \frac{\langle A \cdot B \rangle_S}{(\langle A^2 \rangle_S \cdot \langle B^2 \rangle_S)^{1/2}}, \qquad (5)$$

which clearly depends on the averaging interval S, a dependence understood in the sequel. An indication about the relative importance of anisotropy with respect to average curvature fluctuations can be obtained comparing the phase space normalized correlation functions $\tilde{C}[\vartheta, \delta\vartheta]$ and $\tilde{C}[\kappa, \delta\kappa]$, being $\delta A(s) \doteq A(s) - \langle A \rangle$. Using them we build up a quantity which apparently describes only the local geometric features of the submanifold explored (in a lapse S) by the geodesic having initial conditions $[\mathbf{q}(0), \mathbf{p}(0)] = (\mathbf{q}_0, \mathbf{p}_0)$:

$$R_F[S] \equiv R_F(\mathbf{q}_0, \mathbf{p}_0) \doteq \frac{\tilde{C}[\vartheta, \delta\vartheta]}{\tilde{C}[\kappa, \delta\kappa]} \geq 0 \,. \tag{6}$$

However, the inspection of Fig. 1 shows a striking correspondence between the Poincaré surface of section (PSS) and the map of R_F^2. Indeed, Figs. 1(a) and 1(b), which refer to the typical energy value $E = 1/8$, show that the smaller the value of $R_F(\mathbf{q}_0, \mathbf{p}_0)$, then the *more regular* the geodesic passing through $(\mathbf{q}_0, \mathbf{p}_0)$. To obtain the plot of Fig. 1(b), we chose a grid of points $\{y_0, p_{y_0}\}$ on the PSS $x = 0$, choosing p_{x_0} such that $H = E$, and then numerically integrated the geodesic equations, computing the correlation functions entering $R_F(\mathbf{q}_0, \mathbf{p}_0)$ [17]. The same results have been obtained at all the energies; the more *essential,* though less *suggestive,* histograms of Fig. 2, confirm that our GIC is able to depict correctly the single orbit's behavior up to the dissociation energy. The columns in this plot represent the values of $R_{F_E}^2$, defined as

$$R_{F_E}^2(\mathbf{q}_0, \mathbf{p}_0) \doteq \alpha^2(E) R_F^2(\mathbf{q}_0, \mathbf{p}_0) \,, \tag{7}$$

where $\alpha(E) = \alpha_1 \cdot E^2$ is a scaling factor to get rid of a global energy dependence, which is useful to compare results at different energies. For each energy, the $R_{F_E}^2$ values are reported for a sample of seven initial conditions, chosen among those topologically equivalent to the ones indicated in Fig. 1(b) for $E = 1/8$. We observe that while chaotic trajectories are always characterized by $R_{F_E}^2$ values around its upper limit (normalized to unity, this is because we use $R_{F_E}^2$ instead of R_F), regular orbits have instead considerably smaller values, which is, however, higher for those geodesics tending to become chaotic earlier, as the energy increases. In particular, from both the figures, we see that the geodesics in the regular islands located on the p_y axis of the PSS are recognized by our GIC as nearer to chaotic ones than those belonging either to the large regular island on the y axis or to the banana region; this explains why these islands disappear earlier as the energy increases and gives also some insights on the causes of the (partial) failure of the Riemannian approach to describe these peculiar orbits [2,7,10]. Moreover, very interesting insights can be obtained looking at the relaxation patterns of R_F as a function of the averaging time S [9], as can be perceived from the diffusive behavior around the border of regular islands. Well then, a geometric local quantity turns out to be deeply related

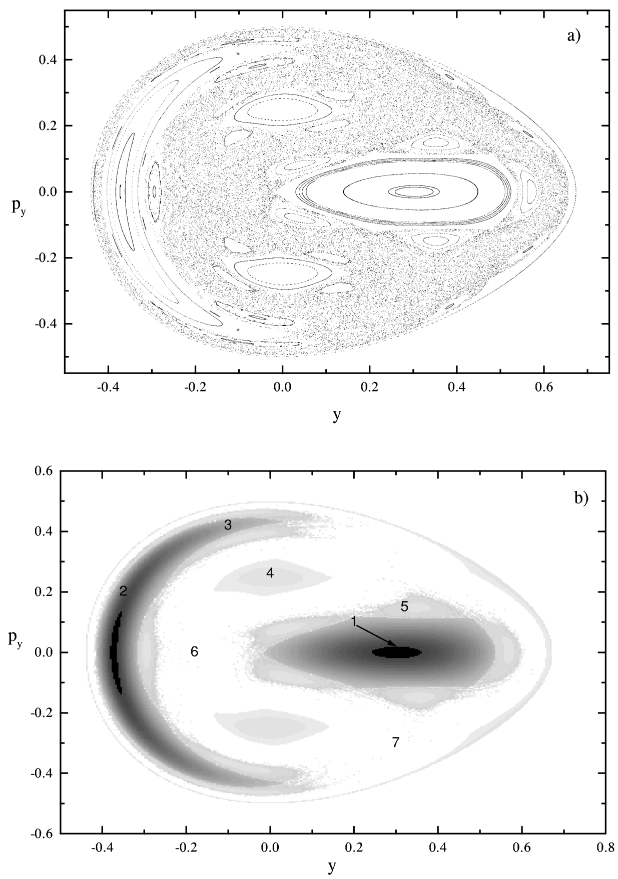

FIG. 1. Phase space portraits of HH system at $E = 1/8$: (a) PSS obtained numerically integrating a sample of orbits up to $T = 10\,000$ units; (b) Gray-scale plot of $R_F^2(y_0, p_{y_0})$ computed following a set of geodesics for a lapse $S = 200$ units. Darker dots represent smaller R_F values, the *white stochastic sea* corresponds to $R_{F_E}^2 > 0.8$. The numbers label the initial conditions of Fig. 2.

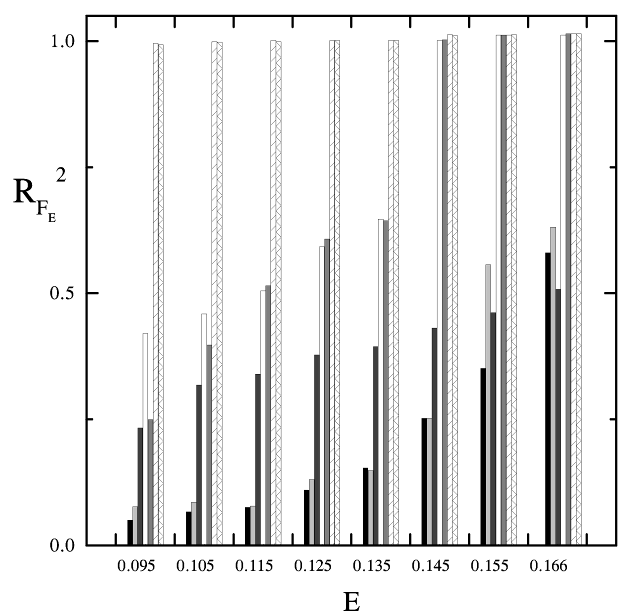

FIG. 2. Values of $R_{F_E}^2$ for geodesics starting from initial conditions "topologically equivalent" to those indicated on Fig. 1 for $E = 1/8$, at energies from $E = 0.095$ up to $E = 0.166$.

to the asymptotic behavior of geodesics. We stress that the values plotted in the map and histograms are obtained through computations of correlation functions over time intervals much shorter than those needed to obtain either the PSS's or the LCN's: it suffices to follow a geodesic for as few as fifteen periods in order to see if the value of R_F attains the upper limit characterizing the stochastic sea or tends to a lower value, which though different for distinct regular orbits, is, however, always smaller than for chaotic ones. A local signature of asymptotic instability acquires a special significance also on the light of the issue of reliability of long time numerical integrations of chaotic systems [18]. The definition of R_F obviously implies that it can be used also as a global indicator, able to give a quantitative measure of the overall degree of stochasticity at a fixed value of energy: a phase-averaged local indicator is a global one; instead, a global indicator is, in general, nonlocal. Perplexities which can be raised by the apparently cumbersome definition of R_F are answered on the light of the pathologies affecting the HH Hamiltonian, which amount mainly to the degeneracies of its integrable limit (see p. 46 in [19]). Indeed, for most two dimensional DS's, more naïve Finsler geometric indicators suffice to discriminate between chaotic and regular orbits. Although the outcomes presented here (and in [2,9]) clearly support the reliability of Finsler GDA, nevertheless a better understanding is still waiting, as we need to test and extend the proposed criterion to more general and higher dimensional DS's. Moreover, it is desirable to improve the rather phenomenological interpretation of R_F and possibly to predict theoretically the threshold above which stochasticity occurs. This goal amounts essentially to understanding whether high R_F values either follow from or are a cause of stochastic behavior (or both). The results obtained support the conviction [2,5–8] that negative curvature is unnecessary (if not irrelevant) to explain the onset of chaos in realistic DS's, keeping in mind, however, that some faded global correlations exist. Instead, a separate spot concerns the long-lasting claims about the implications of scalar curvature: except for two dimensional or isotropic manifolds, it appears to have nothing to do with the behavior of geodesics representing realistic DS's [8] and a hopefully coherent explanation of its irrelevance will be presented elsewhere. Among the issues still open, it is worth investigating more deeply the relevance of the rotation of eigendirections of $\mathcal{H}$: a direct inspection of the JLC equations seems to indicate that the rotation of the perturbation vector should give a contribution to instability; nevertheless a separation by hand of *interacting effects* often causes serious inconsistencies.

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[10] One of the outstanding outcomes of the investigations of Pettini and co-workers resides in the possibility of finding indicators of chaos through microcanonical averages, without integrations of trajectories. It is an obvious price to pay for this achievement to renounce, in principle, to study the local behavior of the DS, looking only to its average global features. The indicator found in [7] fails indeed to distinguish between regular and stochastic orbits on a given energy surface: it assumes in chaotic regions of the phase space a value in between those taken in two (disjoint) regular islands [9]. This is confirmed in [2] where a thorough exploration of the HH phase space reveals somewhere quantitative (and even qualitative) disagreements between dynamical behavior and Riemannian description.
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