Random Walks and Quantum Gravity in Two Dimensions

Bertrand Duplantier

Service de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette Cedex
and Institut Henri Poincaré, 11, rue Pierre et Marie Curie, F-75231 Paris Cedex 05, France

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We consider L planar random walks (or Brownian motions) of large length t, starting at neighboring points, and the probability $P_L(t) \sim t^{-\zeta_L}$ that their paths do not intersect. By a 2D quantum gravity method, i.e., a nonlinear map to an exact solution on a random surface, I establish our former conjecture that $\zeta_L = \frac{1}{24}(4L^2 - 1)$. This also applies to the half plane where $\tilde{\zeta}_L = \frac{L}{3}(1 + 2L)$, as well as to nonintersection exponents of unions of paths. Mandelbrot's conjecture for the Hausdorff dimension $D_H = 4/3$ of the frontier of a Brownian path follows from $\zeta_{3/2}$. [S0031-9007(98)07852-1]

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Random walks (RW) and their continuum counterpart, Brownian motions (BM), describe perhaps the simplest stochastic process pervading statistical physics, probability theory, and even biology. However, quantum mechanics and, more importantly, interacting quantum field theory can be described very generally in terms of the statistics of Brownian paths and of their intersections [1]. This equivalence, used in polymer theory [1] and in rigorous studies of second-order phase transitions and field theories [2–4], suggests that the geometrical properties of random walks are inherent to the quantum world. In probability theory also, nontrivial properties of Brownian paths have led to intriguing conjectures. Mandelbrot [5] suggested that in two dimensions, the hull or external frontier of a planar Brownian path has a Hausdorff dimension $D_H = 4/3$, identical to that of a planar polymer. Families of universal critical exponents are associated with intersection properties of sets of random walks [6–8]. We show here that in two dimensions these exponents can be derived from conformal invariance methods involving quantum gravity. This establishes former conjectures [7], including the Brownian frontier conjecture above, and hints at deep connections between two apparently remote fields, probability theory and string field theory.

Consider a number L of independent random walks (or Brownian paths) $B^{(l)}$, $l = 1, \dots, L$ in $\mathbb{Z}^d$ (or $\mathbb{R}^d$), starting at fixed neighboring points, and the probability $P_L(t) = P\{\cup_{l,l'=1}^{L}(B^{(l)}[0,t] \cap B^{(l')}[0,t]) = \emptyset\}$ that the intersection of their paths up to time t is empty [3,6]. At large times and for $d < 4$, one expects this probability to decay as $P_L(t) \sim t^{-\zeta_L}$, where $\zeta_L(d)$ is a *universal* exponent depending only on L and d. Above the upper critical dimension $d = 4$, RW's almost surely do not intersect. The existence of exponents ζ_L in $d = 2, 3$ and their universality have been proven [8], and they can be calculated near $d = 4$ by renormalization theory [6]. In *two dimensions* (2D), a generalization was introduced [7] for L walks constrained to stay in a half plane, and starting at neighboring points on the boundary, with a nonintersection probability $\tilde{P}_L(t)$ of their paths governed by a "surface" critical exponent $\tilde{\zeta}_L$ such that $\tilde{P}_L(t) \sim t^{-\tilde{\zeta}_L}$.

We have conjectured from conformal invariance arguments and numerical simulations that in 2D [7]

$$\zeta_L = h_{0,L}^{(c=0)} = \frac{1}{24}(4L^2 - 1), \tag{1}$$

and for the half plane

$$2\tilde{\zeta}_L = h_{1,2L+2}^{(c=0)} = \frac{1}{3}L(1 + 2L), \tag{2}$$

where $h_{p,q}^{(c)}$ denotes the Kač conformal weight

$$h_{p,q}^{(c)} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \tag{3}$$

of a minimal conformal field theory of central charge $c = 1 - 6/m(m+1)$, $m \in \mathbb{N}^*$ [9]. For Brownian motions $c = 0$, and $m = 2$. For $L = 1$, the intriguing $\zeta_1 = 1/8$ is actually the disconnection exponent governing the probability that the origin of a single walk remains accessible from infinity without crossing the walk.

Many two-dimensional statistical systems like the Ising model, $O(n)$, and Potts models have been described by height models and Coulomb gas techniques [10], yielding exact values of critical exponents, in agreement with the conformal invariance classification [9,11]. However, the conjectures (1) and (2), even if numerically confirmed [12], and bearing on perhaps the most natural conformally invariant system, seemed to stay out of reach.

This Letter provides the main lines of a derivation of these exponents. The idea is to put the random walks on a random lattice with planar geometry, or, in other words, in the presence of two-dimensional *quantum gravity* [13,14]. There, the conformal dimensions of nonintersecting walks can be obtained from an exact solution. We then use the nonlinear map which exists between conformal dimensions on a random surface and in the plane [14], to obtain results (1) and (2). Generalizations of these exponents are obtained, from which the Brownian frontier dimension $4/3$ follows.

Random surfaces, in relation to string theory [15], have been the subject and source of important developments in statistical methanics in two dimensions. In particular, the discretization of string models led to the consideration of abstract random lattices G, the connectivity fluctuations of which represent those of the metric, i.e., pure 2D quantum gravity [16]. One can then put any 2D statistical model (like Ising model [17], polymers [18]) on the random planar graph G, thereby obtaining a new critical behavior, corresponding to the confluence of the criticality of the random surface G with the critical point of the original model. The critical system "dressed by gravity" has a larger conformal symmetry which allowed Knizhnik, Polyakov, and Zamolodchikov (KPZ) [14] to derive the (amazing) relation between the conformal dimensions $\Delta^{(0)}$ of scaling operators in the plane and those in presence of gravity, Δ: $\Delta^{(0)} = \Delta[1 - (1 - \Delta)/\kappa]$, where κ is a parameter related to the central charge of the statistical model in the plane: $c = 1 - 6(1 - \kappa)^2/\kappa$; for a minimal model of the series (3), $\kappa = 1 + 1/m$, and $\Delta^{(0)}_{p,q} \equiv h^{(c)}_{p,q}$.

Let us now consider as a statistical model *random walks* on a *random graph*. We know [7] that their central charge $c = 0$, whence $m = 2$, $\kappa = 3/2$. Thus the KPZ relation becomes

$$\Delta^{(0)} = \frac{1}{3}\,\Delta(1 + 2\Delta)\,, \tag{4}$$

which bears a striking resemblance to our conjecture (2).

Consider the set of planar random graphs G, built up here with, e.g., trivalent vertices and with a fixed topology, that of a sphere (S) or a disk ($\mathcal{D}$). The partition function is defined as

$$Z_\chi(\beta) = \sum_G \frac{1}{S(G)}\, e^{-\beta|G|}, \tag{5}$$

where χ denotes the Euler characteristic $\chi = 2(S)$, $1(\mathcal{D})$; $|G|$ is the number of vertices of G, $S(G)$ its symmetry factor. The partition sum converges for all values of the parameter β larger than some critical β_c. At $\beta \to \beta_c^+$, a singularity appears due to the presence of infinite graphs in (5)

$$Z_\chi(\beta) \sim (\beta - \beta_c)^{2-\gamma_{\rm str}(\chi)}, \tag{6}$$

where $\gamma_{\rm str}(\chi)$ is the string susceptibility exponent. For pure gravity as described in (5), the embedding dimension $d = 0$ coincides with the central charge $c = 0$, and $\gamma_{\rm str}(\chi) = 2 - \frac{5}{4}\chi$ [19]. The restricted partition function of a planar random graph with the topology of a disk and a fixed number n of external vertices reads

$$G_n(\beta) = \sum_{n-\text{leg planar } G} e^{-\beta|G|}, \tag{7}$$

and can be calculated through the large-N limit of a random $N \times N$ matrix integral [20]. It has an integral representation

$$G_n(\beta) = \int_a^b d\lambda\,\rho(\lambda)\lambda^n, \tag{8}$$

where $\rho(\lambda)$ is the spectral eigenvalue density of the random matrix, for which the explicit expression is known as a function of λ, β [20]. The support of the spectral density $[a,b]$ depends on β.

Now, imagine putting a set of L random walks $B^{(l)}$, $l = 1,\ldots,L$ on the *random graph* G with the special constraint that they start at the same vertex $i \in G$, end at the same vertex $j \in G$, and have no intersections in between. Consider the set $B^{(l)}[i,j]$ of the points visited on the random graph by a given walk $B^{(l)}$ between i and j, and for each site $k \in B^{(l)}[i,j]$ the first entry; i.e., the edge of G along which the walk (l) reached k for the first time. The union of these edges form a tree $T^{(l)}_{i,j}$ spanning all the sites of $B^{(l)}[i,j]$, called the forward tree. An important property is that the measure on all the trees spanning a given set of points visited by a RW is *uniform* [21]. This means that we can also represent the path of a RW by its spanning tree taken with uniform probability. Furthermore, the nonintersection property of the walks is by definition equivalent to that of their spanning trees. Thus we are led to introduce the L-tree partition function on the random lattice

$$Z_L(\beta,z) = \sum_{\text{planar } G} \frac{1}{S(G)}\, e^{-\beta|G|} \sum_{i,j\in G} \sum_{\substack{T^{(l)}_{ij} \\ l=1,\ldots,L}} z^{|T|}, \tag{9}$$

where $\{T^{(l)}_{ij}, l = 1,\ldots,L\}$ is a set of L trees, all constrained to have sites i and j as end points, and without mutual intersections; a fugacity z is in addition associated with the total number $|T| = |\cup_{l=1}^{L} T^{(l)}|$ of vertices of the trees. In principle, the trees spanning the RW paths can have divalent or trivalent vertices on G, but this is immaterial to the critical behavior, as is the choice of purely trivalent graphs G, so we restrict ourselves here to trivalent trees.

The partition function (9) has been calculated exactly in a previous work [18]. The twofold grand canonical partition function is calculated first by summing over the abstract tree configurations, and then gluing patches of random lattice in-between these trees. A tree generating function is defined as $T(x) = \sum_{n\geq 1} x^n T_n$, where $T_1 \equiv 1$ and T_n is the number of *rooted* planar trees with n external vertices (excluding the root). It reads [18]

$$T(x) = \frac{1}{2}\,(1 - \sqrt{1 - 4x})\,. \tag{10}$$

The result for (9) is then given by a multiple integral

$$Z_L(\beta,z) = \int_a^b \prod_{l=1}^{L} d\lambda_l\,\rho(\lambda_l) \prod_{l=1}^{L} \mathcal{T}(z\lambda_l, z\lambda_{l+1})\,, \tag{11}$$

with the cyclic condition $\lambda_{L+1} \equiv \lambda_1$. The geometrical interpretation is quite clear (Fig. 1). Each patch $l = 1,\ldots,L$ of random surface between trees $T^{(l-1)}$, $T^{(l)}$ contributes as a factor a spectral density $\rho(\lambda_l)$ as in Eq. (8), while the backbone of the boundary tree $T^{(l)}$ contributes an inverse "propagator" $\mathcal{T}(z\lambda_l, z\lambda_{l+1})$, which

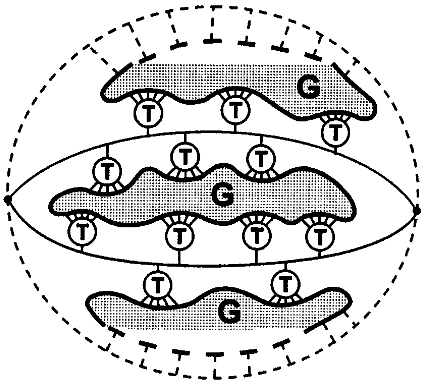

FIG. 1. Random trees on a random surface. The shaded areas represent portions of random lattices G with a disk topology [generating function (7,8)]; $L = 2$ trees connect the end points, each branch giving a generating function T (10). Two possible topologies are represented: for the disk, the dashed lines represent the boundary, whereas they should be discarded for the sphere, the upper and lower grey patches being identified.

couples the adjacent eigenvalues λ_l, λ_{l+1}:

$$\mathcal{T}(x,y) \equiv [1 - T(x) - T(y)]^{-1}. \tag{12}$$

We generalize this to the *boundary* case where G now has the topology of a disk and where the trees connect two sites i and j now on the boundary ∂G:

$$\tilde{Z}_L(\beta, z, \tilde{z}) = \sum_{\text{disk } G} e^{-\beta|G|} \tilde{z}^{|\partial G|} \sum_{i,j \in G} \sum_{\substack{T_{ij}^{(l)} \\ l=1,\ldots,L}} z^{|T|}, \tag{13}$$

where $\tilde{z}$ is the fugacity associated with the boundary's length. The integral representation is

$$\tilde{Z}_L(\beta, z, \tilde{z}) = \int_a^b \prod_{l=1}^{L+1} d\lambda_l \rho(\lambda_l) \prod_{l=1}^{L} \mathcal{T}(z\lambda_l, z\lambda_{l+1}) \times (1 - \tilde{z}\lambda_1)^{-1}(1 - \tilde{z}\lambda_{L+1})^{-1} \tag{14}$$

with two extra propagators describing the two boundary segments.

The critical behavior of the double grand canonical partition function $Z_L(\beta, z)$ (11) associated with nonintersecting RW's on a random lattice is then obtained by taking the double scaling limit $\beta \to \beta_c$ (infinite random surface) and $z \to z_c$ (infinite trees on RW's). The latter is obtained for the smallest z where $\mathcal{T}(z\lambda_l, z\lambda_{l+1})$ (12) vanishes. This occurs near the upper edge of the support of $\rho(\lambda)$, i.e., when $\lambda \to b^-$; thus for $4 z_c b(\beta_c) = 1$ [see (10)]. Hereafter we note $\beta - \beta_c \equiv \delta\beta$, and $z_c - z \equiv \delta z$. For $\lambda \to b^-$ and for $\delta\beta \to 0$, one knows that ρ has the singular behavior (up to constant coefficients) [20] $\rho \sim (\delta\beta)^{1/2}(b - \lambda)^{1/2} + (b - \lambda)^{3/2}$. By homogeneity, each integration of density ρ yields a singular power behavior $\int d\lambda\rho \sim (\delta\beta)^{1/2}(\delta z)^{3/2} + (\delta z)^{5/2}$, while each propagator $\mathcal{T}$ (10), (12) brings in a square root singularity $\mathcal{T} \sim (\delta z)^{-1/2}$. We therefore arrive at a formal power behavior $Z_L \sim [(\delta\beta)^{1/2}\delta z + (\delta z)^2]^L$. The analysis of this singular behavior in terms of conformal dimensions is best performed by using *finite size scaling* (FSS) [18]. One balances the two terms of Z_L above against each other so that $\delta z \sim (\delta\beta)^{1/2}$; i.e., $|T| \sim |G|^{1/2}$, whence

$$Z_L(\beta, z) \sim (\beta - \beta_c)^L. \tag{15}$$

Z_L (11) represents a random surface with two punctures where two conformal operators of dimension Δ_L are located (here two vertices of L nonintersecting RW's), and in a graphical way scales as

$$Z_L \sim Z[\odot]\,|G|^{-2\Delta_L} = \frac{\partial^2}{\partial\beta^2} Z_{\chi=2}(\beta)\,|G|^{-2\Delta_L}, \tag{16}$$

where the random surface partition function Z_χ is given by (6). Equation (16) immediately gives

$$2\Delta_L - \gamma_{\text{str}}(\chi = 2) = L\,, \tag{17}$$

where $\gamma_{\text{str}}(\chi = 2) = -1/2$.

For the boundary partition function $\tilde{Z}_L$ (14) a similar analysis can be performed near the triple critical point $[\beta_c, z_c, \tilde{z}_c = 1/b(\beta_c)]$, where the boundary length also diverges. To extract the conformal dimensions, a *triple* scaling limit requires the further equivalence $\tilde{z}_c - \tilde{z} \sim \delta z \sim (\delta\beta)^{1/2}$, as obvious from homogeneity in (14), i.e., $|\partial G| \sim |G|^{1/2} \sim |T|$. The boundary partition function $\tilde{Z}_L$ corresponds to two boundary operators of dimensions $\tilde{\Delta}_L$, integrated over ∂G, on a random surface with the topology of a disk, or in graphical terms:

$$\tilde{Z}_L \sim Z(\bigcirc)\,|\partial G|^{-2\tilde{\Delta}_L} \sim Z_{\chi=1}(\beta)\,|\partial G|^{2-2\tilde{\Delta}_L}. \tag{18}$$

We can use the punctured disk partition function

$$Z(\bigcirc) = \int_a^b d\lambda \rho(\lambda)\,(1 - \tilde{z}\lambda)^{-2},$$

to divide (14) by the above; comparing to (11), we get

$$\tilde{Z}_L / Z(\bigcirc) \sim \left(\int d\lambda\rho\right)^L \mathcal{T}^L \sim Z_L \sim (Z_1)^L, \tag{19}$$

where the equivalences hold true in terms of scaling behavior. Comparing Eqs. (16) and (19), and using the FSS $|\partial G| \sim |G|^{1/2}$ gives the general identity between surfaces and bulk exponents:

$$\tilde{\Delta}_L = 2\Delta_L - \gamma_{\text{str}}(\chi = 2) = L\,. \tag{20}$$

Applying the quadratic KPZ relation (4) to Δ_L and $\tilde{\Delta}_L = L$ of Eq. (20) yields at once the values in the plane $\mathbb{R}^2$, $\Delta_L^{(0)} \equiv \zeta_L$ [Eq. (1)], and $\tilde{\Delta}_L^{(0)} \equiv 2\tilde{\zeta}_L$ [Eq. (2)], QED.

Equation (20) gives the key to many generalizations. Indeed the product of propagators $\mathcal{T}^L$ there can be replaced by a product $\prod_l \mathcal{T}_l$ corresponding to different geometrical objects, as obvious from the construction (see Fig. 1). Consider then the generalizations of exponents $\zeta(n_1, \ldots, n_L) = \Delta^{(0)}\{n_l\}$, as well as $2\tilde{\zeta}(n_1, \ldots, n_L) = \tilde{\Delta}^{(0)}\{n_l\}$, describing L mutually avoiding bunches $l = 1, \ldots, L$, each made of n_l walks *transparent* to each other [22]. In the presence of gravity, each bunch will contribute a certain inverse propagator $\mathcal{T}_{n_l}$ and yield instead of (20)

$$Z\{n_l\} \sim \frac{\tilde{Z}\{n_l\}}{Z(\bigcirc)} \sim \left(\int d\lambda\rho\right)^L \prod_{l=1}^{L} \mathcal{T}_{n_l}\,, \tag{21}$$

to be identified with $|\partial G|^{-2\tilde{\Delta}\{n_l\}}$. The *factorization* property (21) immediately implies the *additivity of boundary conformal dimensions in the presence of gravity*

$$\tilde{\Delta}\{n_1,\ldots,n_L\} = \sum_{l=1}^{L} \tilde{\Delta}(n_l)\,, \tag{22}$$

where $\tilde{\Delta}(n)$ is now the boundary dimension of a *single* bunch of n transparent walks on the random surface. We know $\tilde{\Delta}(n)$ exactly since it corresponds in the standard plane to a trivial surface conformal dimension $\tilde{\Delta}^{(0)}(n) = n$. It thus suffices to *invert* (4) to get

$$\tilde{\Delta}(n) = \tfrac{1}{4}\left(\sqrt{24n+1} - 1\right). \tag{23}$$

Because of the identification in (21) of the bulk partition function with the ratio of boundary ones, we also have

$$2\Delta\{n_1,\ldots,n_L\} + \tfrac{1}{2} = \tilde{\Delta}\{n_1,\ldots,n_L\}.$$

In the plane, using once again the KPZ relation (4) for $\Delta\{n_l\}$ and $\tilde{\Delta}\{n_l\}$ gives the general result

$$\begin{aligned} \zeta(n_1,\ldots,n_L) &= \tfrac{1}{24}(4x^2 - 1), \\ 2\tilde{\zeta}(n_1,\ldots,n_L) &= \tfrac{1}{3}x(1+2x) \\ x &= \sum_{l=1}^{L} \tilde{\Delta}(n_l)\,. \end{aligned} \tag{24}$$

Recently, Lawler and Werner [23] proved by purely probabilistic means that there exist two (unknown) functions F and U such that $\zeta\{n_l\} = F(x)$ and $2\tilde{\zeta}\{n_l\} = U(x)$, where $x = \sum_l U^{-1}(n_l)$, and U^{-1} denotes the inverse function of U. Quantum gravity methods here explain this structure in terms of linear equation (22), and give the explicit functions $F(x)$ and $U(x)$ of (24), together with $U^{-1}(n) \equiv \tilde{\Delta}(n)$ in (23).

Let us finish by remarking that (24) yields for $\zeta(2, 1^{(L)})$ describing a two-sided walk and L one-sided walks, all mutually nonintersecting, $\zeta(2,1^{(L)}) = \zeta_{L+3/2}$. For $L = 1$, $\zeta(2,1) = \zeta_{5/2} = 1$ gives correctly the escape probability of a RW from another RW. For $L = 0$, $\zeta(2,1^{(0)}) = \zeta_{3/2} = 1/3$ is the two-sided disconnection exponent. It is related to the Hausdorff dimension of the frontier by $D = 2 - 2\zeta$ [24]. Thus we derive $D_H = 4/3$, i.e., Mandelbrot's conjecture.

Obviously, the robust quantum geometric structure explicited here allows many generalizations.

W. Werner's talk at the Institut Henri Poincaré [23] provided a motivation for reanalyzing this problem.

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