Self-Organized Criticality in a Mixed Hierarchical System

M. G. Shnirman and E. M. Blanter
International Institute for Earthquake Prediction Theory and Mathematical Geophysics, Warshavskoye sh 79, korp 2, Moscow 113556, Russia
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It is demonstrated that the self-organized criticality emerges in a hierarchical system as a result of heterogeneity in the scaling conditions. A heterogeneous system consisting of a mixture of the simplest hierarchical models of failure is considered. Four types of model behavior are obtained: stability, catastrophe, unstable criticality, and stable (self-organized) criticality. It is shown that the self-organized criticality reflects heterogeneous properties of media. Possible applications to the study of seismicity are discussed. [S0031-9007(98)07823-5]

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Self-organized critical phenomena have drawn close attention since their discovery by Bak, Tang, and Wiesenfeld [1]. They constructed an avalanche system (sandpile model) which exhibited a critical behavior, reflected in the linear form of the magnitude-frequency relationship, and not associated with any phase transition. This kind of behavior was designated by them as *self-organized criticality* (SOC). The linear form of the magnitude-frequency relationship was observed in various natural and artificial systems (see references cited in [1]). In particular, in geophysics it has been known for a long time as the Gutenberg-Richter law [2]; the idea of SOC was widely discussed in this context [3–8]. Systems demonstrating self-organized criticality are widely applied to the modeling of seismic processes [3,9–15] and are able to reproduce various basic features of seismicity.

The main interest in the field since the paper [1] is to investigate the conditions after which SOC emerges. A number of models exhibiting SOC were proposed; among them avalanche systems [3,11], hierarchical systems of defect development [10,14–16], or the combination of both types [13]. It was discovered that SOC in hierarchical systems may appear as a consequence of nonmonotone conditions of destruction [16] or feedback relations [10,15]. These results, however, do not explain the fact that SOC is a common feature of nature. Such an explanation could be given provided SOC generally emerges as a consequence of some inherent property of natural phenomena.

The importance of heterogeneity in relation to critical phenomena was previously established [17]. In this Letter, we investigate the role of heterogeneity in the emergence of SOC. We consider a simple hierarchical system with heterogeneous conditions of destruction, and show that appearance of SOC reflects the degree of heterogeneity. Since heterogeneity is a common feature of natural systems, one can expect that SOC is, indeed, typical for the evolution of inhomogeneous media, such as the lithosphere of the Earth. In addition, we identify other possible regimes of evolution of the system: stability, catastrophe, and unstable criticality. We investigate their emergence in dependence on heterogeneity involved in the system.

Description of the model.—We consider a hierarchical system of elements with branching number $n = 3$ (Fig. 1). Each element at the level $l + 1$ corresponds to a group of three elements of the previous level l. Elements of the system have two possible states, broken or unbroken. An element in a broken state is referred to as a defect.

States of elements at the level $l + 1$ are determined by the number of defects in relevant groups of three elements at the previous level l. The number of defects in the group sufficient to obtain a defect at the superior level is referred to as the critical number. Any configuration of defects in a group of three elements, containing a critical number or more defects, is referred to as a critical configuration.

In previous models [10,14–16,18] destruction conditions were constant for all elements of the system. In the present model the critical number k may be different for different elements ($k = 1, 2, 3$). It is assumed that the critical number is independently determined, by a random choice, for each element of the system. The fraction of elements with the critical number k is denoted as a_k. We refer to the parameters a_k as concentrations of the mixture; they determine heterogeneity of the system. It is

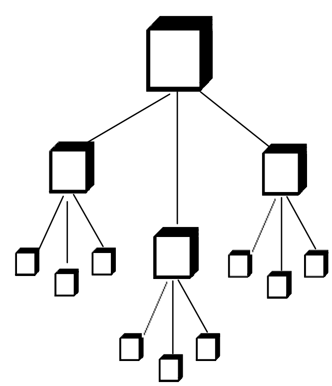

FIG. 1. A hierarchical system with the branching number $n = 3$.

assumed that concentrations a_k are invariant with respect to the level number l, reflecting self-similarity of conditions of destruction. The sum of the concentrations is equal to unity ($a_1 + a_2 + a_3 = 1$); therefore it is necessary to define only two from three concentrations, for example, a_1 and a_2. Homogeneous systems correspond to degenerated cases of the mixture, when one of the concentrations a_k is equal to unity and the two others are zero.

The density of defects at the level $l + 1$ can be expressed as follows:

$$p(l + 1) = F[p(l)], \tag{1}$$

where $F(p)$ denotes the probability to obtain a critical configuration of defects in a group of three elements, if the probability of a defect is equal to p. Assumption of self-similarity implies that the transition function F is the same for all levels of the system. The behavior of the system is governed by the transition function F and the initial density of defects at the first level of the hierarchy $p(1)$. The density of critical configurations, containing exactly k defects in a group of three elements at level l is equal to $V_k = C_3^k p^k (1 - p)^{3-k}$, where $p = p(l)$ is the density of defects at level l. Then, the density of critical configurations, containing k or more defects in a group, is equal to $W_k = \sum_{l=k}^{3} V_l$. This implies the following form for the function $F(p)$:

$$\begin{aligned} F(p) &= \sum_k a_k W_k(p) \\ &= 3a_1 p(1 - p)^2 + 3(a_1 + a_2)p^2(1 - p) + p^3, \end{aligned} \tag{2}$$

which is completely determined by the concentrations a_k.

Magnitude-frequency relation and criticality.—In studies of seismicity the magnitude of an earthquake is actually used as a measure of the energy of the earthquake. A linear relation between the magnitude of the earthquake and the linear size of its source is established [19]:

$$\log_{10} S \approx M + \text{const}. \tag{3}$$

In the present model, following Refs. [16], we consider the magnitude as a characteristic of the size of a defect at level l:

$$M(l) = l \log_{10} 3. \tag{4}$$

Expressing the average number of defects at the level l,

$$N(l) = 3^{L-l} p(l), \tag{5}$$

the magnitude-frequency relation for our model, which is an analog of Eq. (3) for seismicity, reads as

$$\log_{10} N(l) = -M(l) + \log_{10} p(l) + \text{const}. \tag{6}$$

We associate critical behavior with a linear form of the magnitude-frequency relation with a slope equal to unity (cf. [1]). This definition of criticality means that the critical behavior may be obtained when densities of defects $p(l)$ tend to a constant value $p_0 > 0$ with the growth of the level number l [see Eq. (6)]. A simple form of the transition function (2) allows one to determine parametric areas relevant to different types of system behavior, to describe areas where the critical behavior may be observed, and to estimate parameters of the mixture necessary to obtain self-organized criticality.

Parameters of the mixture and system behavior.—The behavior of the densities of defects $p(l)$ with growth of the level number l is governed by concentrations a_k of the mixture and the initial density $p(1)$. Now we investigate behavior of the densities $p(l)$ for different parameters of the mixture a_k. We consider the transition function $F(p)$ for fixed parameters of the mixture a_k. The function $F(p)$ is continuous and smooth. It follows from Eq. (2) that the map defined by the function $F(p)$ has three fixed points: $p = 0$, $p = 1$, and $p = p_0 = (1 - 3a_1)/(3a_2 - 1)$. The derivation $dF/dp = 3a_1$ for $p = 0$; $dF/dp = 3a_3$ for $p = 1$; and $dF/dp = 1 - (1 - 3a_1)(1 - 3a_3)/(1 - 3a_2)$ for $p = p_0$. The behavior of densities $p(l)$ depends on stable and unstable fixed points of the map $F(p)$ inside the interval $(0, 1)$, which are governed by the corresponding derivation dF/dp.

(i) Stability.—When the fixed point p_0 is either below zero or above unity, only two fixed points ($p = 0$ and $p = 1$) exist inside the interval $(0, 1)$. The function $F(p)$ lies below the diagonal line (Fig. 2a) or above it (Fig. 2b).

In the first case $F(p) < p$ for all values p (Fig. 2a). Then it follows from Eq. (1) that the densities $p(l)$ tend to zero with the growth of the level number l (Fig. 3a).

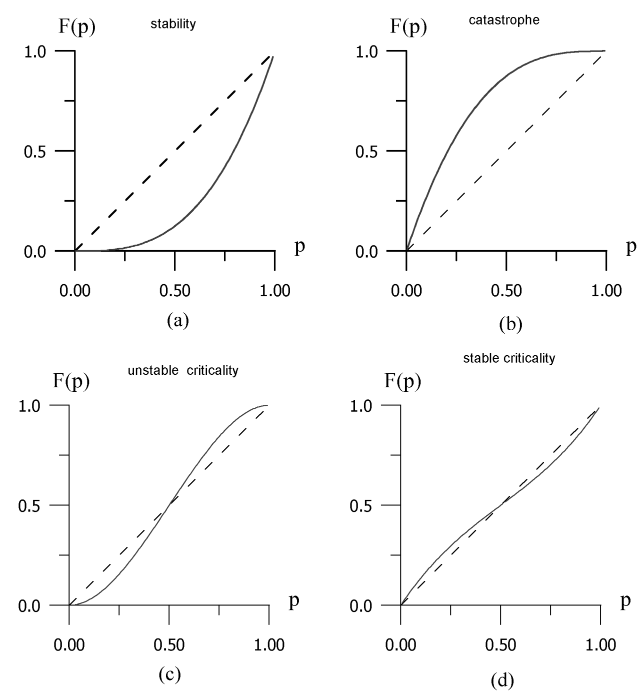

FIG. 2. The transition function $F(p)$ for different parametric areas. (a) Area of stability; (b) area of catastrophe; (c) area of unstable scale invariance; (d) area of stable scale invariance.

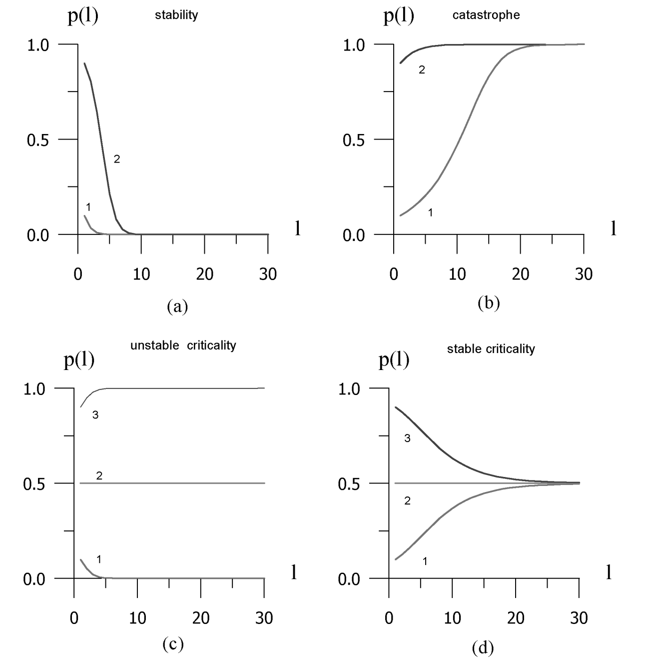

FIG. 3. Densities of defects for different parametric areas. (a) Area of stability ($a_1 = 0.1$, $a_2 = 0.2$): $p(1) = 0.1$ for curve 1 and $p(1) = 0.9$ for curve 2; (b) area of catastrophe ($a_1 = 0.4$, $a_2 = 0.4$): $p(1) = 0.1$ for curve 1 and $p(1) = 0.9$ for curve 2; (c) area of unstable criticality ($a_1 = 0.1$, $a_2 = 0.8$): $p(1) = 0.1$ for curve 1, $p(1) = 0.5$ for curve 2, and $p(1) = 0.9$ for curve 3; (d) area of stable criticality ($a_1 = 0.45$, $a_2 = 0.1$): $p(1) = 0.1$ for curve 1, $p(1) = 0.5$ for curve 2, and $p(1) = 0.9$ for curve 3.

We denote this kind of behavior as *stability,* since the density of defects at the highest level L is very small $p(L) \approx 0$, and a perturbation actually does not reach high levels of the system. This kind of behavior appears under the following conditions (Fig. 4):

$$a_1 < 1/3, \qquad a_1 + a_2 < 2/3 . \tag{7}$$

(ii) Catastrophe.—When $F(p)$ lies above the diagonal line (Fig. 2b), or equivalently $F(p) > p$ for any p, the densities $p(l)$ tend to unity with the growth of the level number l (Fig. 3b). The density of defects of the highest level $p(L)$ is close to unity, and thus high levels of the system are completely destroyed. Therefore this kind of behavior is denoted as *catastrophe*. It occurs for the following concentrations of the mixture a_k (Fig. 4):

$$a_1 > 1/3, \qquad a_1 + a_2 > 2/3 . \tag{8}$$

(iii) Unstable criticality.—When the fixed point p_0 is inside the interval (0, 1), for $p(1) = p_0$ all of the densities of defects assume the same value $p(l) = p_0$. It follows from Eq. (6) that the magnitude-frequency relation in this case is linear with a slope equal to unity. This means that the system exhibits critical behavior.

When concentrations of the mixture a_k satisfy the conditions $a_1 < 1/3$ and $a_1 + a_2 > 2/3$ simultaneously, the fixed point p_0 is unstable (Fig. 2c). For $p < p_0$, one has $F(p) < p$. Thus, for all $p(1) < p_0$ the densities of defects $p(l)$ tend to zero with the growth of l (Fig. 3c, curve 1). For $p > p_0$ one obtains $F(p) > p$, and for all $p(1) > p_0$ the densities of defects $p(l)$ tend to unity with the growth of l (Fig. 3c, curve 3). Thus, critical behavior exists in the single unstable point p_0 (Fig. 3c, curve 2). We denote this kind of behavior as *unstable criticality* (Fig. 4).

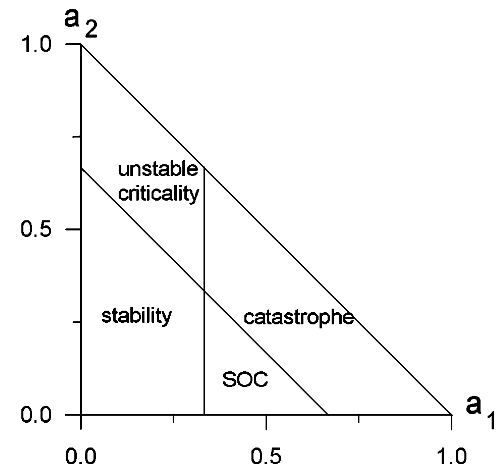

FIG. 4. Different parametric areas of behavior of the system. The triangle is divided by two straight lines: $a_1 = 1/3$ and $a_1 + a_2 = 2/3$.

(iv) Stable criticality (SOC).—When concentrations of the mixture a_k satisfy the conditions $a_1 > 1/3$ and $a_1 + a_2 < 2/3$ at the same time, the fixed point p_0 is stable (Fig. 2d). For all values of the initial density $p(1)$ the densities of defects $p(l)$ tend to the value p_0 with the growth of the level number l (Fig. 3d). The critical behavior exists for all values of initial density of defects $p(1)$, and may be denoted as *stable* or *self-organized criticality* (Fig. 4).

Figure 4 shows all possible types of behavior of the system. Three of them—stability, catastrophe, and unstable criticality—may be obtained for homogeneous rules of destruction, when the critical number k is the same for all elements of the system ($k = 3$, 1, and 2, respectively). In contrast, self-organized criticality occurs only under heterogeneous conditions of destruction: the most different rules of destruction (elements corresponding to critical numbers $k = 1$ and $k = 3$) must be mixed with sufficiently high concentrations ($a_1 > 1/3$ and $a_3 > 1/3$). This condition determines high heterogeneity of the system with self-organized critical behavior.

Despite the fact that the considered model is not a dynamic one and does not describe the temporal evolution of any natural system, it has a wide field of applications. It may be applied to the description of processes, where the result is more important than details of evolution, such as the rupture process. It also reflects statistic properties of stationary processes.

The described model allows for combination of both stable and unstable criticality in the behavior of one system. This combination may be very important in order to understand the nature of different predictability of catastrophic events, in particular, that of strong earthquakes. Applications of a fixed precursor of a catastrophic event give different results in models with unstable [20] and stable [21] critical behavior. Variations of predictability for a fixed algorithm were observed in the earthquake prediction [22]. The complex behavior obtained in the present simple model allows one to obtain variations of the predictability of strong events and to investigate the validity of various precursors of a strong earthquake in different parametric areas.

In conclusion, it was shown that heterogeneity of the medium is crucial for the appearance of self-organized critical behavior. The existence of a phase diagram in which SOC appears only for sufficiently large heterogeneity has been announced for a different system in [23]. Here, we demonstrate that SOC really emerges when the heterogeneity of destruction conditions is relatively strong; the most robust and the most fragile elements must be mixed in large proportions ($a_1 \geq 1/3$, $a_3 \geq 1/3$). Thus the self-organized criticality (linear form of the magnitude-frequency relationship) is expected to be a common feature of various systems with high heterogeneity.

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