Hopf Bifurcation from Chaos and Generalized Winding Numbers of Critical Modes

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In the study of chaos, Lyapunov exponents have been successfully used in describing the expansion and contraction rates of various modes. In this Letter, generalized winding numbers are defined in association with the corresponding Lyapunov exponents to characterize the rotation behavior of these modes during the evolution. A Hopf bifurcation from chaos, namely, a blowout bifurcation with certain finite typical frequency, is revealed. The frequency of the motion after the bifurcation is justified to be equal to the generalized winding number of the critical transverse mode, for which the Lyapunov exponent crosses zero at the bifurcation. [S0031-9007(98)07798-9]

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In the past several decades, the investigation of chaotic behavior has assumed a central role in nonlinear science. The study of the Lyapunov exponent spectrum is one of the foundations of chaos analysis [1,2]. Lyapunov exponents represent the important behavior of average expansion (or contraction) rates of various typical modes around a chaotic attractor. Another very important feature, the average rotation rate of these modes, has been considered much less. In Refs. [3,4], the authors considered the influence of external noise on Hopf bifurcations of nonchaotic deterministic systems, and rotation rates were computed in association with Lyapunov exponents. Recently, the investigation of the average phase frequency of chaos (which is, actually, the winding number of chaotic trajectories of continuous dissipative systems) has become a very interesting problem [5–8]. In this Letter, we will generalize these ideas to consider the average phase frequencies of various modes, which will be called generalized winding numbers (GWN), corresponding to different Lyapunov exponents of chaotic systems. These GWNs take the winding number studied so far as its special case (the winding number in Refs. [5–8] is identified to the GWN of the special mode of zero Lyapunov exponent).

Another interesting problem in the chaos study is the topological change of chaos with variation of a system parameter. Recently, a new bifurcation (namely, blowout bifurcation leading to on-off intermittency), characterized by the largest transverse Lyapunov exponent crossing zero, has attracted much attention [9,10]. However, in this study, only the transverse Lyapunov exponent is considered, and the rotation behavior (or, say, winding number) of the unstable transverse mode has escaped from attention. We find that this rotation feature, described by the corresponding GWN, is very important for characterizing the behavior of the on-off intermittent state just after the blowout bifurcation. Specifically, we find a typical Hopf bifurcation from chaos, after which a regular motion is born from chaos, whose frequency is determined by the GWN of the critical transverse mode of the reference chaotic state.

First, let us consider the Lorenz model;

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= xy - \beta z. \end{aligned} \tag{1}$$

At $\sigma = 10$ and $\beta = 1$, the system is in the chaotic region for $\rho > 24$ (of course, many periodic windows can be found in this region). In Fig. 1(a), we plot the three Lyapunov exponents of the chaotic attractor vs ρ, with λ_1, λ_3 characterizing the expansion and contraction modes, respectively, and $\lambda_2 = 0$ representing the mode along the trajectory. The modes corresponding to various λ_i, $i = 1, 2, 3$, can be defined by the standard method well-known in computing Lyapunov spectrum. Here, we go further to specify the average rotations, i.e., the GWNs, of these modes. The computation of these average frequencies can be performed in the same way as [5–8]. We can either compute the zero crossings of a given variable, e.g., Δx_i or Δy_i, or compute the average frequency of an angle, such as $\theta_i(t), \tan(\theta_i) = \Delta y_i/\Delta x_i$, where Δx_i and Δy_i are the infinitesimal orbit deviations of the ith mode corresponding to the ith Lyapunov exponent λ_i. Thus, the GWN $\langle \omega_i \rangle$ is defined as the average frequency,

$$\langle \omega_i \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \dot{\theta}_i(t)\, dt. \tag{2}$$

In computation, we use very large T for approximation.

The above computations of phase frequencies are exactly the same as those used for the conventional phase problem [5–8]. However, a key difference is that we are computing the winding numbers of various modes corresponding to all Lyapunov exponents, while the winding number computed until now dealt with the trajectory itself only, corresponding to the single mode of $\lambda = 0$.

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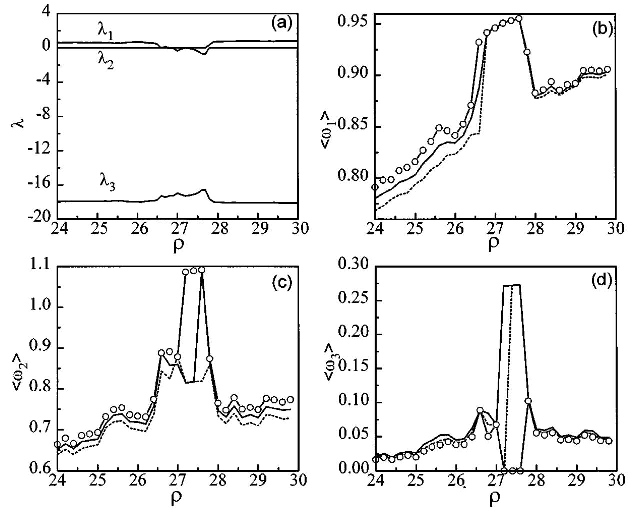

FIG. 1. (a) The three Lyapunov exponents of the Lorenz model (1) plotted vs ρ. $\sigma = 10$ and $\beta = 1$ (which will be used for all of the following figures). Some period windows are not shown due to the rough resolution of the parameter changes. (b)–(d) The three generalized winding numbers of (2) corresponding to the three Lyapunov exponents of λ_1, λ_2, and λ_3, respectively. The solid, dashed, and circled lines are computed by counting the zero crossings of Δx_i and Δy_i, and computing $\dot{\theta}_i(t)$ with $\tan\theta_i = \Delta y_i/\Delta x_i$, respectively. All three curves are identical within the computation precision in the chaos region.

In Figs. 1(b), 1(c), and 1(d), we plot $\langle\omega_i\rangle$, $i = 1, 2, 3$, vs ρ, corresponding to the λ_i, $i = 1, 2, 3$, respectively. The following interesting points can be observed. (i) The results obtained by computing the zero crossings of Δx_i, Δy_i, and the average frequency of $\theta_i(t)$ with $\tan(\theta_i) = \Delta y_i/\Delta x_i$ are identical to each other within the computing precision in the chaos region. (ii) The conventional winding number computed in early papers is just $\langle\omega_2\rangle$ in our case, corresponding to $\lambda_2 = 0$. (iii) Generalized winding numbers corresponding to negative exponents seem to be useless, and have been ignored in all previous papers. However, as a bifurcation from chaos takes place, some of these winding numbers become of crucial importance for predicting the behavior of the state after the bifurcation; this is the focus of the following part of the present Letter.

Now let us consider the coupled Lorenz systems:

$$\begin{aligned}
\dot{x}(j) &= \sigma[y(j) - x(j)], \\
\dot{y}(j) &= \rho x(j) - y(j) - x(j)z(j) \\
&\quad + (\epsilon + r)[x(j+1) - x(j)] \\
&\quad + (\epsilon - r)[x(j-1) - x(j)], \\
\dot{z}(j) &= x(j)y(j) - \beta z(j), \\
j &= 1, 2, \ldots, N,
\end{aligned} \tag{3}$$

where the periodic boundary condition $\vec{u}(j + N, t) = \vec{u}(j, t), \vec{u}(t) = (x(t), y(t), z(t))^T$, is applied. ϵ and r represent the diffusive and gradient couplings, respectively. We keep $\rho = 28, \epsilon = 14, N = 4$, and have r changed, and study the topological change of the system state. For small r, we have synchronous chaos, $\vec{u}(j, t) = \vec{s}(t)$, $j = 1, 2, 3, 4$, where $\vec{s}(t)$ is a chaotic orbit of the single Lorenz model. As r increases past a critical value $r_c = 7.945$, the synchronization breaks. The first four Lyapunov exponents of the coupled systems are plotted in Fig. 2(a), where the first two are the same as those of the single site Lorenz model before bifurcation since all of the sites are synchronized to each other. The $\lambda_{3,4}$ (a long, while elementary, computation can show that $\lambda_{3,4}$ correspond to the first space mode with wave number $k = 1$ having $N = 4$ as its space period) can be changed by varying r. At $r = r_c = 7.945$, $\lambda_{3,4}$ (i.e., the largest transverse Lyapunov exponents of the given synchronous chaotic state) come up to zero, indicating a kind of blowout bifurcation. In Fig. 2(b), we plot $\langle\omega_i\rangle, i = 1, 2, 3, 4$ vs r. It seems that $\langle\omega_{3,4}\rangle$ ($\langle\omega_3\rangle = \langle\omega_4\rangle$) are not interesting before the bifurcation, since the third and fourth modes damp exponentially by the rate $e^{\lambda_{3,4}t}$ during the system evolution, and do not affect the asymptotic behavior of the synchronous chaotic state. However, this quantity is of key importance in determining the dynamic feature of the system after the bifurcation, as we will see later.

In Fig. 3(a), we plot $x(1, t) - x(2, t)$ vs t at $r = 7.95$, which is slightly larger than r_c. While desynchronization appears, the variation clearly shows certain characteristic frequency. In Figs. 3(b) and 3(c), we plot the spectrum of $x(1, t)$ right before and after the instability of the reference synchronous state. It is interesting to find a sharp peak in Fig. 3(c) (which is absent before the instability) exactly at $\langle\omega_{3,4}\rangle$ of Fig. 2(b). In Fig. 3(d), we eliminate the component of synchronous chaos, and plot the spectrum of $x(1, t) - x(2, t)$, where the spectrum peak at $\langle\omega_{3,4}\rangle$ is very clearly manifested. All of the facts (i) double identical Lyapunov exponents crossing zero at the critical point in Fig. 2(a); (ii) the two equal frequencies (the generalized winding numbers now) $\langle\omega_3\rangle = \langle\omega_4\rangle$ existing before the bifurcation; (iii) the clear

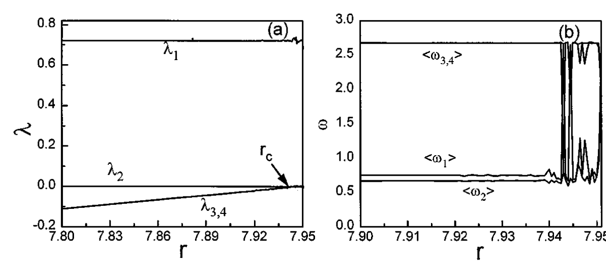

FIG. 2. (a) The first four Lyapunov exponents of the coupled Lorenz systems (3) plotted vs r. $N = 4$, $\epsilon = 14$, and $\rho = 28$. For $r > r_c = 7.945$ (at $r_c, \lambda_{3,4}$ come up to zero), the synchronous chaos loses its stability. (b) The generalized winding numbers corresponding to the Lyapunov exponents of (a) obtained by computing the average rotating frequency of $\theta_i(t)$ with $\tan(\theta_i) = \Delta y_i(j = 1)/\Delta x_i(j = 1)$.

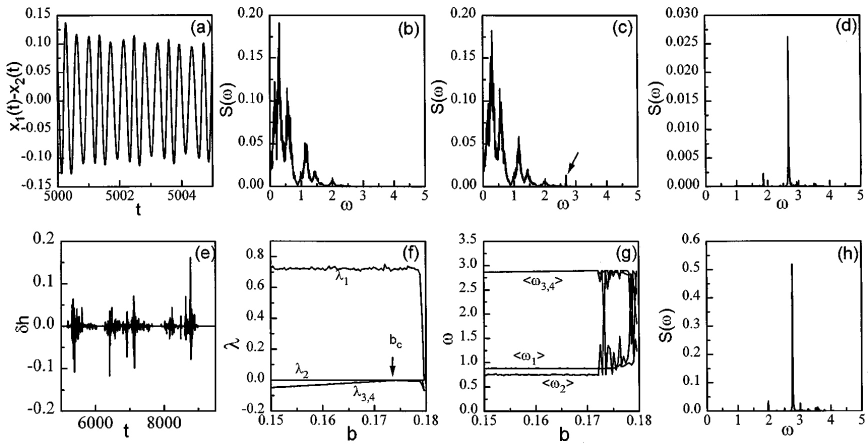

FIG. 3. (a) $x(1,t) - x(2,t)$ plotted vs t at $r = 7.95$ slightly larger than r_c. Desynchronization appears, and the variation contains certain regular oscillation clearly. (b) The spectrum of $x(1,t)$ at $r = 7.93$ slightly smaller than r_c. (c) The spectrum of $x(1,t)$ at $r = 7.95$ slightly larger than r_c. In (c), a new sharp peak far away from the continuous chaotic spectrum of (b) appears, which has frequency exactly equal to the generalized winding numbers $\langle \omega_{3,4} \rangle$ in Fig. 2(b). (d) The spectrum of $x(1,t) - x(2,t)$ at $r = 7.95$. The characteristic frequency is shown in a clearer way. (e) $\delta h = \{[x(1) + x(3)] - [x(2) + x(4)]\}/2$ plotted vs t. Both the unstable $k = 1$ mode and the synchronous chaos are eliminated by the plus and minus operations, respectively. A typical on-off intermittency is identified for the remaining irregular desynchronous components. (f)–(h) The same as Figs. 2(a), 2(b), and 3(d), respectively, where the nonlinear coupling (5) is applied. $\epsilon = 14$, $r = 7.5$ for (f), (g), and (h); and $b = 0.178 > b_c = 0.174$ for (h). Hopf bifurcation from chaos is again identified for the nonlinear coupling case.

delta-function-like spectrum after the bifurcation and (iv) the accordance of the delta-peak frequency with the GWN of the critical mode before the bifurcation [compare the peak in Fig. 3(d) with the $\langle \omega_{3,4} \rangle$ in Fig. 2(b)] show clearly a Hopf bifurcation from chaos. In Ref. [11], the authors found an unusually fast oscillation mode coming from synchronous chaos, the bifurcation mechanism was, however, not clear. This mode is nothing but a wave with spatial wave number $k = 1$ and time oscillation frequency of $\langle \omega_{3,4} \rangle$, appearing after the Hopf bifurcation from chaos at $r = r_c$.

In Fig. 3(d), we find that the spectrum of $x(1,t) - x(2,t)$ contains some irregular components besides the periodic motion of frequency $\langle \omega_{3,4} \rangle$. Since the unstable mode is $k = 1$, $x(1)$ and $x(3)$ [also $x(2)$ and $x(4)$] must have opposite phases for the unstable mode. Therefore, in $x(1) + x(3)$ and $x(2) + x(4)$ the component of the unstable mode can be effectively ruled out. Moreover, the synchronous chaos can be canceled in $[x(1) + x(3)] - [x(2) + x(4)]$. Therefore, the quantity

$$\delta h = \{[x(1) + x(3)] - [x(2) + x(4)]\}/2 \tag{4}$$

can well represent the component in the desynchronous part of motion apart from the unstable mode. In Fig. 3(e), we plot δh vs t; a typical on-off intermittency is identified. We believe that an oscillation with well-defined frequency together with random on-off intermittency are what we can observe after a Hopf bifurcation from chaos.

In Eqs. (3), we use linear couplings. It is emphasized that the above features of Hopf bifurcation from chaos are robust against the perturbations of nonlinear couplings. For instance, we can multiply the coupling in the second equation of (3) by a factor

$$1 + \frac{b[x(j)^2 + y(j)^2 + z(j)^2]}{1 + [x(j)^2 + y(j)^2 + z(j)^2]}, \tag{5}$$

and treat b as a bifurcation parameter. The results observed are essentially the same as in the linear coupling case [see Figs. 3(f),3(g), and 3(h)].

In summary, we have found an interesting Hopf bifurcation from chaos, of which the characteristic frequency and the wave number (they represent the temporal and spatial orders of the inhomogeneous component of the state after bifurcation) are identified to the generalized winding number and the wave number of the critical transverse modes at the blowout bifurcation, respectively. However, some open problems still exist. First, the computation of the generalized winding numbers shows very large fluctuations in the on-off intermittency region. Second, one often computes the average phase frequency of a chaotic trajectory by analyzing a single observable. It is known that the frequencies computed from different observables may be different. To overcome this difficulty, some "natural" observables are suggested to represent the phase structure of the system [5,8]. Fortunately, one can easily find these natural observables for some well-known chaotic systems, such as Lorenz equations, Rossler equations, Duffing system, and so on. However, it turns out to be difficult to find these natural observables for more complicated high-dimensional chaos, e.g., spatiotemporal chaos. This difficulty exists also for the computation of generalized winding numbers. Therefore, these problems are open for further investigation.

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