

Do Naked Singularities Generically Occur in Generalized Theories of Gravity?

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A new mechanism for causing naked singularities is found in an effective superstring theory. We investigate the gravitational collapse in a spherically symmetric Einstein-Maxwell-dilaton system in the presence of a pure cosmological-constant “potential,” where the system has no static black hole solution. We show that once gravitational collapse occurs in the system, naked singularities necessarily appear in the sense that the field equations break down in the domain of outer communications. This suggests that in generalized theories of gravity, the nonminimally coupled fields generically cause naked singularities in the process of gravitational collapse if the system has no static or stationary black-hole solution. [S0031-9007(98)07856-9]

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The singularity theorem [1] states that the occurrence of singularities is inevitable under some physical conditions in general relativity. There are two notable scenarios where singularities may appear in our Universe. One is the initial singularity at the birth of the Universe and the other is the final stage of gravitational collapse. In the latter case we believe that an event horizon is formed which encloses all occurring singularities as the collapse proceeds, following the cosmic censorship hypothesis (CCH) [2]. CCH is classified into two types, the weak cosmic censorship hypothesis (WCCH) and the corresponding strong one (SCCH). WCCH says that observers at an infinity should not see singularities while SCCH says that no observer should see them and the whole region of a space-time can be uniquely determined by initial regular data. Mathematically, SCCH is equivalent to the statement that no Cauchy horizon can be formed in a physical gravitational collapse.

Recently many elegant results [3] suggest that SCCH holds for charged and/or rotating black holes due to the destruction of the Cauchy horizon by the mass inflation phenomenon. The general proof of CCH is, however, far from complete and many counterexamples have been found in the framework of general relativity [4]. A new approach has been considered in the context of generalized theories of gravity. Gibbons and Maeda [5] discovered the static black-hole solutions in the Einstein-Maxwell-dilaton system, which comes from an effective superstring theory, and later Garfinkel, Horowitz, and Strominger [6] showed that the inner (Cauchy) horizon in the Reissner-Nordström solution is replaced by a spacelike singularity. This suggests that the occurrence of the inner horizon is not generic and hence SCCH holds if we take the effect of string theory into account. On the other hand Horne and Horowitz [7] obtained the opposite result that extremal electrically charged nonstatic black-hole solutions in the presence of a central charge have timelike singularities. It seems, however, not to be a counterexample of SCCH in the sense that such solutions

have no regular initial spacelike hypersurface because of the central singularities. Thus, the following question naturally arises. Does CCH really hold in generalized theories of gravity? There is not much evidence deciding the matter yet because the above results do not take any physical process of the gravitational collapse from initial regular data into account.

Although a large number of studies have been made on finding static/stationary black-hole solution and investigating their thermodynamical properties and geometrical stability and so on in generalized theories of gravity, only few studies have so far been made on investigating gravitational collapse. We should not overlook that the gravitational collapse in generalized theories of gravity will be much different from that in general relativity. In general relativity, the black-hole no-hair conjecture states that the matter fields are swallowed by a black hole and a space-time asymptotically approaches the electrovacuum Kerr-Newman solution after gravitational collapse. On the other hand, the above aspects are not necessarily satisfied in generalized theories of gravity because the dilaton field couples to the curvature and/or other matter fields.

In this Letter we investigate the gravitational collapse in the spherically symmetric Einstein-Maxwell-dilaton system analytically in the presence of a pure positive cosmological constant, which corresponds to $g_1 = 0$ in the Liouville-type potential [8]. This system is interesting because it has been proved that no static black-hole solution exists for spherically symmetric space-times [9]. Such property seems to be general in the sense that there exists no asymptotically flat, asymptotically de Sitter, or asymptotically anti-de Sitter solution in the arbitrary exponential dilaton potential [8]. Furthermore, it is known that there is a critical mass below which no asymptotically flat black-hole solution exists in the system which includes the Gauss-Bonnet term which was neglected in the former works [10,11]. In these systems the following three interpretations are available: (i) The

matter fields do not cause the gravitational collapse but escape to infinity without forming a black-hole event horizon (BEH); (ii) the space-time does not approach a stationary space-time and the matter fields oscillate forever or, (iii) naked singularities are necessarily formed. If the case (i) is true for all initial data, we should find that the strong cosmic no-hair conjecture holds in the present system. By investigating numerically, however, we confirmed that this is not the case and some initial data lead to the gravitational collapse, namely, a trapped surface forms [12]. This result is consistent with the fact that the dilaton field satisfies the dominant energy condition in the Einstein frame. Then it is enough to consider only the cases (ii) and (iii). Case (ii) seems incompatible with our naive expectation that after gravitational collapse, the space-time settles down to an asymptotically stationary space-time with a black hole. If case (iii) is true, we must say that CCH is violated even in generalized theories of gravity.

Using the double null coordinates, we will show that the field equations necessarily break down in the domain of outer communications or at the BEH, once the gravitational collapse has occurred. It is worth noting that “break down” does not imply a coordinate singularity because our coordinates are, probably, the most extended coordinates in spherically symmetric space-times. It is also supported by numerical calculations which show the formation of naked singularities in the system [12]. Thus, our result suggests that naked singularities generically occur in the gravitational collapse of spherically symmetric systems.

The low-energy effective action (Einstein frame) of string theory as a dilaton model is

$$S = \int d^4x \sqrt{-g} [-R + 2(\nabla\phi)^2 + e^{-2\phi}F^2 + 2\Lambda], \tag{1}$$

where R is the Ricci scalar, ϕ is a massless dilaton field, Λ is a positive constant, and $F_{\mu\nu}$ is the field strength of the Maxwell field. The double-null coordinates for a spherically symmetric space-time are

$$ds^2 = -2e^{-\lambda}(U, V)dUdV + R(U, V)^2d\Omega, \tag{2}$$

where ∂_U and ∂_V are future-ingoing and outgoing null geodesics, respectively. The Maxwell equation is automatically satisfied for a purely magnetic Maxwell field $F = Q \sin\theta d\theta \wedge d\phi$ ($F^2 = 2Q^2/R^4$), where Q is the magnetic charge and \wedge is Cartan’s wedge product. An electrically charged solution is obtained by a duality rotation from the magnetically charged one [5,6]. Therefore we shall consider only a purely magnetic case in this Letter. The dynamical field equations are

$$\lambda_{,UV} - \frac{2R_{,UV}}{R} = 2\phi_{,U}\phi_{,V} + e^{-\lambda} \left(\frac{Q^2 e^{-2\phi}}{R^4} - \Lambda \right), \tag{3}$$

$$R_{,UV} + \frac{R_{,U}R_{,V}}{R} = -\frac{e^{-\lambda}}{2R} \left(1 - \frac{Q^2 e^{-2\phi}}{R^2} - \Lambda R^2 \right), \tag{4}$$

$$2R^3(R_{,U}\phi_{,V} + R\phi_{,UV} + R_{,V}\phi_{,U}) = Q^2 e^{-2\phi-\lambda}, \tag{5}$$

where $A_{,a}$ is a partial derivative of A with respect to a . The constraint equations are

$$R_{,UU} + \lambda_{,U}R_{,U} = -(\phi_{,U})^2 R, \tag{6}$$

$$R_{,VV} + \lambda_{,V}R_{,V} = -(\phi_{,V})^2 R. \tag{7}$$

We consider the evolution of the field equations with initial regular data on a null characteristic hypersurface whose boundary is a closed future-trapped surface (see Fig. 1) because we are interested in the gravitational collapse. As shown in Theorem 1 of Ref. [13], the trapped surface causes the formation of a BEH ($U = U_B$ null hypersurface) if there are no singularities observed from I^+ . For this initial value problem, we will show the following theorem.

Theorem.—Let us consider the dynamical evolution of the model (1) in a spherically symmetric space-time or equivalently for Eqs. (3)–(7) with initial data on the characteristic null hypersurface N . Then, there is $U_1 (\leq U_B)$ such that the system of equations breaks down at $U = U_1$

We shall show the above theorem by contradiction below. First, we will consider the asymptotic behavior of field functions near the cosmological event horizon (CEH), which is defined as a past Cauchy horizon $H^-(I^+)$ when a BEH exists (see Ref. [14]). It is convenient to rescale the coordinate U such that U is an affine parameter of a null geodesic of the CEH, i.e., λ is constant along the CEH. Hereafter we use a character $u (= f(U))$ instead of U for parametrization to avoid confusion. Under such coordinates, Eq. (6) on the CEH is

$$-R_{,uu} = (\phi_{,u})^2 R. \tag{8}$$

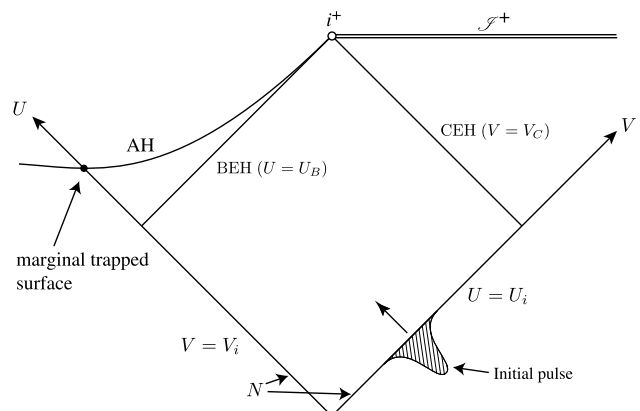


FIG. 1. A Penrose diagram in asymptotically de Sitter space-time. N and AH are the characteristic hypersurface and the apparent horizon of the black hole, respectively.

If there are no singularities in $\overline{J^-(I^+) \cap J^+(N)}$, the null geodesic generators of the BEH and the CEH are future complete. Then, by the nondecreasing area law for the CEH and by existence of an upper bound of the area [14], $\lim_{u \rightarrow \infty} R = C_1$ [hereafter, $C_i (i = 1, 2, \dots)$ means a positive constant], and hence future asymptotic behavior of $R_{,u}$ on the CEH is represented as follows [15]:

$$R_{,u} \sim C_2 u^{-\alpha-1} \quad (\alpha > 0). \quad (9)$$

Then, by Eq. (8)

$$\phi_{,u} \sim u^{-\alpha/2-1}, \quad (10)$$

and hence $\lim_{u \rightarrow \infty} \phi(u) = \text{const}$. We shall obtain the asymptotic value of $R_{,V}$ on the CEH by solving Eq. (4) as

$$R_{,V} = \left(\int_{u_i}^u \frac{KR}{R_i} du + R_{,V}|_i \right) \frac{R_i}{R}, \quad (11)$$

where $R_{,V}|_i$, R_i are initial values of $R_{,V}$, R on $u = u_i$, respectively, and $K = -(1 - Q^2 e^{-2\phi}/R^2 - \Lambda R^2)/2Re^\lambda$. K must approach a positive constant K_∞ as $u \rightarrow \infty$ because R , $\phi \rightarrow \text{const}$ and the expansion $\theta_+ \equiv R_{,V}/R$ of each outgoing null geodesic is positive (if it were negative, the area element $d\Omega$ would become 0 along the outgoing null geodesics). Then, the asymptotic value of $R_{,V}$ is

$$R_{,V} \sim K_\infty u > 0. \quad (12)$$

We shall also obtain the asymptotic value of $\phi_{,V}$ on the CEH by solving Eq. (5) with the following solution:

$$\phi_{,V} = \left(\int_{u_i}^u \frac{HR}{R_i} du + \phi_{,V}|_i \right) \frac{R_i}{R}, \quad (13)$$

where $\phi_{,V}|_i$ is an initial value of $\phi_{,V}$ on $u = u_i$ and $H = (Q^2 e^{-2\phi-\lambda} - 2R^3 R_{,V} \phi_{,u})/2R^4$. H approaches a positive constant H_∞ as $u \rightarrow \infty$ because $R_{,V} \phi_{,u} \sim u^{-\alpha/2} \rightarrow 0$ by Eqs. (10) and (12). The asymptotic value of $\phi_{,V}$ is

$$\phi_{,V} \sim H_\infty u > 0. \quad (14)$$

Next, let us consider an infinitesimally small neighborhood \mathcal{U}_C of the CEH which contains a timelike hypersurface T_C such that T_C is in the past of the CEH by $\epsilon (> 0)$, where ϵ is a fixed affine parameter distance along the outgoing null geodesic intersecting the CEH [16]. We denote each point of the intersection of T_C and $u = \text{const}$ hypersurface by $p(u)$. By Eq. (5), the solution of $h \equiv \phi_{,u}$ along each $u = \text{const}$ is

$$h = \left(- \int_V^{V_C} \frac{LR}{R_C} dV + h_C \right) \frac{R_C}{R}, \quad (15)$$

where R_C and h_C are values of R and h at the CEH ($V = V_C$), respectively, and $L = (Q^2 e^{-2\phi-\lambda} - 2R^3 R_{,u} \phi_{,V})/2R^4$. Since $(R_{,u} \phi_{,V}/R)|_{V_C} \sim u^{-\alpha} \rightarrow 0$ for

large values of u , $L \sim L_\infty > 0$ asymptotically. Differentiating h by V in Eq. (15),

$$h_{,V} = L + \frac{R_C R_{,V}}{R^2} \left(\int_V^{V_C} \frac{LR}{R_C} dV - h_C \right). \quad (16)$$

By the relation $\epsilon = dr \sim udV$ for large u [17], $h_{,V} \sim L_\infty + O(\epsilon) + O(u^{-\alpha/2})$. Since ϵ is an arbitrary small value, asymptotically $h_{,V} > 0$ on the $u = \text{const}$ null segment $[V|_{p(u)}, V_C] \subset \mathcal{U}_C$.

In the next step we investigate the behavior of ϕ , $\phi_{,V}$ and R on BEH, just like the CEH case. We rescale V into v such that v is an affine parameter and $\lambda = \text{const}$ on the BEH, while we leave U unchanged. Since the area of the BEH is nondecreasing and also has an upper bound as shown in [13,14], $\lim_{v \rightarrow \infty} R = C_3$, and $R_{,v} \sim C_4 v^{-\beta-1}$ ($\beta > 0$). This means that by Eq. (7) $\phi_{,v} \sim v^{-\beta/2-1}$ as obtained in the CEH case. By replacing u by v in the argument of the CEH case and solving Eqs. (4) and (5), the asymptotic values of $\phi_{,U}$ and $R_{,U}$ become $R_{,U} \sim v$ and $\phi_{,U} \sim C_5 v$, respectively.

Hence the first and third terms in the left-hand side of the dilaton field Eq. (5) are negligible asymptotically and

$$\lim_{v \rightarrow \infty} k_{,U} = \lim_{v \rightarrow \infty} \phi_{,vU} = C_6 > 0. \quad (17)$$

We define $k \equiv \phi_{,v}$. Consider an infinitesimally small neighborhood \mathcal{U}_B of the BEH. There is a small ϵ such that the timelike hypersurface T_B which is in the past of the BEH by a fixed affine parameter distance ϵ of ingoing null geodesics intersecting the BEH is contained in \mathcal{U}_B . By the relation between the affine parameter and dU , i.e., $dU \propto \epsilon/v$, k on T_B is asymptotically

$$\begin{aligned} k|_{T_B} &\sim k|_{\text{BEH}} + k_{,U}|_{\text{BEH}}(-dU) \\ &\sim k|_{\text{BEH}} - C_6 \epsilon v^{-1} \sim (v^{-\beta/2} - C_6 \epsilon)v^{-1}. \end{aligned} \quad (18)$$

This indicates that $\phi_{,V}$ is negative on T_B for large values $v (> v_1)$. Now, consider each null segment N_u ($u \geq u_I$): $[V_C - \epsilon/u_I, V_C]$ where $h_{,V} > 0$ on N_{u_I} (see Fig. 2). If one takes u_I large enough, N_u intersects T_B ($v > v_1$) at $u = u_F$. Let us take a sequence of N_{u_J} ($J = 1, 2, \dots, L+1$) (L is a natural number large enough), where $\delta u = (u_F - u_I)/L$ and $u_J = u_I + (J-1)\delta u$. Assume that $\phi_{,V} > 0$ on N_J ($\equiv N_{u_J}$), for a moment, then $h_{,V} > 0$ on N_J in the same way as $h_{,V} > 0$ on $[V|_{p(u)}, V_C]$ [18]. Hence we can show that $\phi_{,V}|_{N_{J+1}} \equiv (\phi_{,V}|_{N_J} + h_{,V}|_{N_J} \delta u) > 0$. On the other hand, $\phi_{,V} > 0$ on $u = u_I$ by Eq. (14), hence $\phi_{,V} > 0$ for each u_J by induction. This is a contradiction because as we showed $\phi_{,V} < 0$ on T_B ($u = u_F$).

It is worth commenting that the possibility of the formation of null singularities on BEH, continuing to i^+ is not excluded from our theorem. In this case the BEH is a singular null hypersurface even if WCCH holds (SCCH is, probably, violated).

Our theorem says that the case (ii) is not true, as we expected. When we consider the dyon solution or a rotating space-time, nontrivial 3-rank antisymmetric tensor

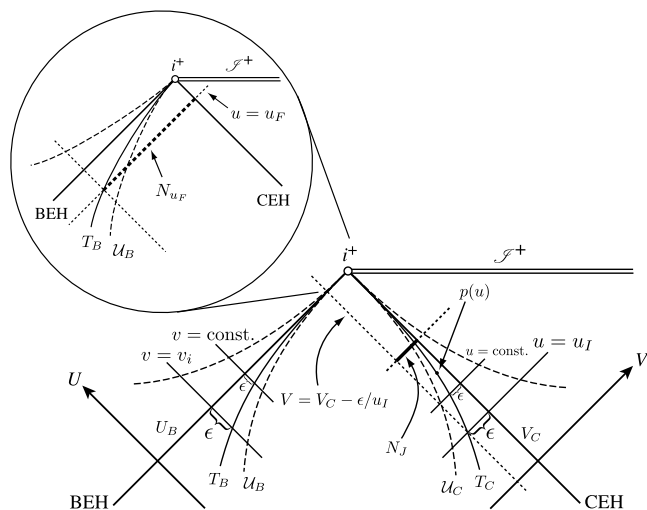


FIG. 2. Timelike hypersurfaces T_B, T_C are displayed in the neighborhoods $\mathcal{U}_B, \mathcal{U}_C$ of BEH and CEH, respectively. A null segment N_J is displayed by a thick line. ϵ is a fixed affine parameter distance along outgoing and ingoing null geodesics.

fields inevitably appear. However, we expect that such fields do not change our result drastically. This implies that if there are no static/stationary black-hole solutions, naked singularities generically occur in the process of gravitational collapse in generalized theories of gravity.

Since the model under consideration is the low energy limit of string theory, we cannot exclude the possibility that higher curvature terms and higher order α' corrections, where α' is the inverse string tension, may prevent the space-time from causing naked singularities. However, the theory is useful near the Planck scale. Thus, we can at least say that *string theory predicts existence of space-time points with very high curvature in the domain of outer communications*. This picture is quite different from that of general relativity because it is strongly believed that space-time points with high curvature do not appear generically in the outer region of a black hole.

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 [15] It is enough to consider the case of the power law function $R_{,u} \sim u^{-\alpha-1}$ to show our theorem because it gives the mildest dumping.
 [16] The neighborhood of CEH can be covered by Gaussian null coordinates, $ds^2 = -2drd\eta + f(d\eta)^2 + R^2d\Omega$, where r is an affine parameter of outgoing null geodesics and $f = 0$ on the CEH. Then, $\epsilon = dr = \text{const}$.
 [17] It seems reasonable to assume that the surface gravity of CEH, $\kappa_c \equiv -f_{,r}/2$, is asymptotically constant. In this case, $e^{\kappa_c \eta} \sim u$ and hence $d\eta \sim du/u$. This indicates that $dr \sim udV$.
 [18] One can show that $R_{,u} < 0$ on T_C for large u by using Eq. (4). Therefore, $R_{,u} < 0$ inside T_C because each expansion of ingoing null geodesics from T_C must decrease monotonically. Thus, we can easily get $h_{,V} > 0$ by considering each segment; $[V_C - \epsilon/u_I, V_C - \epsilon/u_J] \notin \mathcal{U}_C$, $[V_C - \epsilon/u_J, V_C] \in \mathcal{U}_C$. The detailed proof can be found in Ref. [12].