Decoherence of Bose-Einstein Condensates in Traps at Finite Temperature

Robert Graham

Fachbereich Physik, Universität-Gesamthochschule Essen, 45117 Essen, Germany

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The phase diffusion of the order parameter of trapped Bose-Einstein condensates at temperature $k_B T \gg \hbar\bar{\omega}$ is determined, which gives the fundamental limit of the linewidth $\Delta\nu$ of an atom laser. In addition a prediction of the number fluctuations in the condensate and their correlation time τ_c is made and a general relation for $\Delta\nu\tau_c$ is derived from the fluctuation-dissipation relation. [S0031-9007(98)07932-0]

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Bose-Einstein condensation in a weakly interacting Bose gas in three dimensions in the thermodynamic limit of an infinitely extended system is a second order phase transition in which an order parameter, the macroscopic wave function, appears spontaneously with a fixed but arbitrary phase, turning the global U(1) symmetry responsible for particle-number conservation into a broken or hidden symmetry [1]. The rigidity of the phase of the order parameter against local perturbations and the absence of any phase diffusion gives rise to the Goldstone modes, which take the form of collisionless (zero) sound or hydrodynamic sound, respectively, depending on whether the sound frequency is in the collisionless mean-field regime or in the collision-dominated regime [1,2].

In finite systems, and thus also in all trapped Bose gases, sharp phase transitions are impossible and hidden symmetries in a rigorous sense cannot appear [1]. Nevertheless a macroscopic wave function describing a Bose-Einstein condensate (BEC) still exists, as is now firmly established by the experiments [3]. However, for the general reason mentioned, the phase of the macroscopic wave function cannot be stable but must undergo a diffusion process, which restores the U(1) symmetry over sufficiently long time intervals [1]. This diffusion process is therefore different from the Goldstone modes mentioned before, which are oscillations around a fixed value of the phase and do not restore the symmetry [1,2].

Recently a first attempt has been made to measure the stability of the phase of the macroscopic wave function in a trapped BEC. In an experimental setup of considerable ingenuity [4] the relative phase of two BEC's was measured using a time-domain separated oscillatory field condensate interferometer. Over the time interval of 100 ms scanned in the experiment the relative phase was found to be robust. At first sight this experimental result may seem surprising since decoherence of entangled states of many atoms should be extremely rapid. Then, however, one realizes that there is so far no clear theoretical prediction of the decoherence time of Bose-Einstein condensates in traps against which the aforementioned experiment, or extensions of it which will surely follow, could be checked. In a number of papers [5] the dispersion of the phase of a trapped Bose-Einstein condensate at zero temperature was considered, which is due to thermodynamic fluctuations $\delta\mu$ of the chemical potential μ in a finite system with fixed particle number. An extension of this mechanism to finite temperature has also been proposed [6]. This effect is not a "phase diffusion" but corresponds to an effect of inhomogeneous broadening, and is even reversible in "revivals." The experiments are done at "high" temperature $k_B T \gg \hbar\bar{\omega}$ and even $k_B T \gg \mu$, where $\bar{\omega}$ is the geometrical mean of the three main trap frequencies. One would expect a proper phase-diffusion process to occur in such a regime due to the interaction of the condensate with a thermal bath of collective modes and quasiparticles, but so far an understanding of this process seems lacking (see, however, [7]). This gap in our understanding of Bose-Einstein condensates in traps at finite temperature is particularly painful, because the fundamental limit of the linewidth of an atom laser depends on it: there is not yet a "Schawlow-Townes" formula [8] for the linewidth of an atom laser, because its derivation requires a prior understanding of phase diffusion in Bose-Einstein condensates.

In the present paper I outline a theory of dissipation and thermal fluctuations in a trapped Bose-Einstein condensate which is used to determine the phase-diffusion constant, and from it, the linewidth of a trapped Bose-Einstein condensate as a function of temperature. The result obtained explains the experimentally observed robustness of the phase. I find it convenient to present first a phenomenological framework for the theory, in the form of Langevin equations in which dissipation appears via phenomenological parameters and the fluctuation-dissipation relation is invoked to determine the fluctuations. Then the phenomenological parameters are fixed by drawing on known microscopic results for the damping of collective modes and calculating the new transport coefficient in the Langevin equation of the condensate. It is found to result primarily from scattering of thermally excited collective modes (phonons) off the condensate. This coefficient then determines the phase-diffusion constant and the fundamental linewidth of an atom laser via a Schawlow-Townes-type formula.

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The weakly interacting Bose gas in a trap in standard notation is described by the Hamiltonian

$$H = \int d^3x \hat{\psi}^+ \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(\boldsymbol{x}) - \mu + \frac{U_0}{2} \hat{\psi}^+ \hat{\psi} \right\} \hat{\psi} . \tag{1}$$

The total number of atoms N is fixed. μ is the chemical potential. The presence of a Bose-Einstein condensate means that many ($N_0 \gg 1$) particles occupy the normalized single-particle state $\psi_0(\boldsymbol{x})$ of lowest energy μ satisfying [9] $-(\hbar^2/2m)\nabla^2\psi_0 + [V(\boldsymbol{x}) + U_0 N_0 |\psi_0|^2]\psi_0 = \mu\psi_0$.

The number density of the condensate is $n_0(\boldsymbol{x}) = N_0|\psi_0(\boldsymbol{x})|^2$. Both $\psi_0(\boldsymbol{x})$ and N_0 are functions of μ. In the following we turn this around and consider μ a function of N_0. The presence of the highly occupied condensate mode makes the decomposition of the Heisenberg field operator $\hat{\psi}(\boldsymbol{x},t) = [\alpha_0(t)\psi_0(\boldsymbol{x}) + \hat{\psi}'(\boldsymbol{x},t)]e^{-i\mu t/\hbar}$ useful, where we shall follow the tradition starting with Bogoliubov [10] and describe the condensate classically, remembering, however, that $N_0 = |\alpha_0|^2$ is the particle number in the condensate. $\hat{\psi}'(\boldsymbol{x},t)$ is the field operator for the particles outside the condensate. The Hamiltonian then splits according to $H = H_0 + H_1 + H_2 + H_3 + H_4$, where H_n ($n = 0,1,2,3,4$) comprises the terms of H which are of nth order in $\hat{\psi}', \hat{\psi}'^+$, respectively. The equation of motion for the condensate amplitude α_0 receives contributions from H_0 to H_3. For a discussion of various microscopic approximation schemes, see [11].

Quasiparticle operators $\hat{\alpha}_\nu, \hat{\alpha}_\nu^+$ are defined by the standard Bogoliubov transformation $\hat{\psi}'(\boldsymbol{x},t) = \sum_\nu [u_\nu(\boldsymbol{x})\hat{\alpha}_\nu(t) + v_\nu^*(\boldsymbol{x})\hat{\alpha}_\nu^+(t)]$, where u_ν, v_ν satisfy the usual Bogoliubov–de Gennes equations [11], and the $\hat{\alpha}_\nu$ obey the Heisenberg equations of motion $\dot{\hat{\alpha}}_\nu = \frac{i}{\hbar}[\hat{H}^{(\nu)}, \hat{\alpha}_\nu]$ with $\hat{H}^{(\nu)} = \hbar\omega_\nu[\hat{\alpha}_\nu^+\hat{\alpha}_\nu - \int |v_\nu(\boldsymbol{x})|^2 d^3x]$. Again $|u_\nu|^2$, $|v_\nu|^2$, and ω_ν are functions of $|\alpha_0|^2$.

In order to derive a phenomenological equation of motion for the condensate alone we turn to the free energy $F(|\alpha_0|^2)$ of a fluctuation of $|\alpha_0|^2$ from equilibrium. Expanded to second order around its minimum it takes the form

$$\beta F(|\alpha_0|^2) = \frac{(|\alpha_0|^2 - \langle|\alpha_0|^2\rangle)^2}{2\langle\delta N_0^2\rangle}$$

with $\langle|\alpha_0|^2\rangle = N - \langle\hat{N}'\rangle$, $\langle\delta N_0^2\rangle = \langle\hat{N}'^2\rangle - \langle\hat{N}'\rangle^2$, where $\hat{N}' = \int d^3x \hat{\psi}'^+(\boldsymbol{x})\hat{\psi}'(\boldsymbol{x})$. The expectation values $\langle\hat{N}'\rangle$ and $\langle\delta N_0^2\rangle$ (which turns out to be anomalously large $\sim N^{4/3}$) have recently been evaluated within the Bogoliubov theory [12] and can therefore here be considered as known. The equation of motion of α_0 near thermal equilibrium can now be written with the help of $F(|\alpha_0|^2)$ in the general form [13]

$$i\hbar\dot{\alpha}_0 = (1 - i\Gamma_0)\frac{\partial F}{\partial\alpha_0^*} + F_0(t) \tag{2}$$

with Gaussian white noise $F_0(t)$ satisfying

$$\langle F_0(t)\rangle = 0, \qquad \langle F_0^*(t)F_0(0)\rangle = 2\hbar k_B T\Gamma_0\delta(t)$$

determined so as to ensure the correct equilibrium distribution [14] $\rho(\alpha_0,\alpha_0^*) = Z_0^{-1}\exp[-F(|\alpha_0|^2)/k_B T]$ for the condensate. Here I ignore the possibility of the occurrence of squeezing in the thermal bath of uncondensed particles. Then only a single new phenomenological coefficient Γ_0, a dimensionless but surely temperature dependent number, remains to be determined below.

As a short digression let us also extend the equations of motion of the quasiparticles to include dissipation and fluctuation within a phenomenological Markoffian framework [15]. It is convenient to do this by writing quantum Langevin equations [16]

$$i\hbar\dot{\hat{\alpha}}_\nu = (1 - i\Gamma_\nu)\frac{\partial\hat{H}^{(\nu)}}{\partial\hat{\alpha}_\nu^+} + \hat{F}_\nu(t) \tag{3}$$

with Gaussian Langevin-force operators satisfying $\langle\hat{F}_\nu(t)\rangle = 0$, $\langle[\hat{F}_\nu(t), \hat{F}_{\nu'}^+(t')]\rangle = 2\hbar^2\Gamma_\nu\omega_\nu\delta(t-t')\delta_{\nu\nu'}$ and, by the fluctuation dissipation theorem,

$$\langle\hat{F}_\nu^+(t)\hat{F}_{\nu'}(t')\rangle = 2\hbar^2\Gamma_\nu\omega_\nu\bar{n}_\nu\delta(t-t')\delta_{\nu\nu'} \tag{4}$$

with the Planck distribution $\bar{n}_\nu = (e^{\beta\hbar\omega_\nu} - 1)^{-1}$. The phenomenological coefficients Γ_ν have the meaning of one-half of the inverse Q-factor of mode ν, $\Gamma_\nu = (2Q_\nu)^{-1}$, and remain to be determined below.

But let us now consider how number fluctuations and the phase diffusion of the condensate in equilibrium follows from (2). Its deterministic part describes the relaxation of the condensate to the minimum of the free energy F of the condensate at $\langle N_0\rangle = \langle|\alpha_0|^2\rangle$. The particle number fluctuations $\delta N_0 = |\alpha_0|^2 - \langle|\alpha_0|^2\rangle$ in equilibrium, after linearizing (2) in δN_0, are found to have the correlation function

$$\langle\delta N_0(t)\delta N_0(t')\rangle = \langle\delta N_0^2\rangle e^{-|t-t'|/\tau_c} \tag{5}$$

with the correlation time $\tau_c = (\hbar\langle\delta N_0^2\rangle/2\Gamma_0\langle N_0\rangle k_B T)$ which could in principle be measured by taking the Fourier transformation in time of time-resolved *in situ* phase-contrast images of the condensate. On a time scale very much larger than the correlation time τ_c the phase φ_0 of the condensate in equilibrium, i.e., the phase of $\alpha_0 = \sqrt{N_0}\, e^{i\varphi_0}$, satisfies the Langevin equation of a Wiener process with diffusion constant

$$D_\varphi = k_B T(\Gamma_0 + \Gamma_0^{-1})/(\hbar\langle N_0\rangle), \tag{6}$$

i.e., $\langle[\varphi_0(t) - \varphi_0(0)]^2\rangle = D_\varphi t$. The expectation value $\langle\alpha_0(t)\rangle$ then decays exponentially according to $\langle\alpha_0(t)\rangle = \sqrt{\langle N_0\rangle}\, e^{-\Delta\nu t}$ with the linewidth $\Delta\nu$ given by the Schawlow-Townes-type formula

$$\Delta\nu = k_B T(\Gamma_0 + \Gamma_0^{-1})/(2\hbar\langle N_0\rangle) \geq k_B T/(\hbar N_0) . \tag{7}$$

The general relation between $\Delta\nu$ and τ_c,

$$\Delta\nu = \frac{k_B T}{2\hbar\langle N_0\rangle}\left(\frac{\hbar\langle\delta N_0^2\rangle}{2\langle N_0\rangle k_B T\tau_c} + \frac{2\langle N_0\rangle k_B T\tau_c}{\hbar\langle\delta N_0^2\rangle}\right), \tag{8}$$

follows from our phenomenological theory, which is independent of the yet unknown coefficient Γ_0 and holds for the general value of $k_B T$ (outside the critical region). Equation (8) is a general consequence of the fluctuation dissipation relation for the condensate in the absence of squeezing in the bath of uncondensed atoms.

Let us now determine the numbers Γ_0, Γ_ν from microscopic considerations, starting with Γ_ν for $\nu \neq 0$. We shall here confine our attention to the damping of the low-lying collective modes in the collisionless regime, even though our phenomenological framework may still be used in the collision-dominated regime. Furthermore, we confine ourselves to the experimentally relevant regime $k_BT \gg \mu$. The damping of the low-lying collective modes in this regime is due to Landau damping, described by a part of H_3, and was calculated for spatially homogeneous condensates first by Szépfalusy and Kondor [17]. Their result written for our coeficient Γ_ν yields $\Gamma_\nu = A_\nu(k_BT/\mu)(n_0a^3)^{1/2}$. The numerical coefficient A_ν turns out to be independent of ν in the spatially homogeneous system, and its value there is [18] $A_\nu = 3\pi^{3/2}/4$. However, a similar expression for Γ_ν was even shown to hold for the collective excitations in traps [19] where n_0 is the condensate density $n_0(\mathbf{0})$ in the center of the trap and A_ν is a numerical coefficient which depends on the trap geometry and the mode function for mode ν. For the dipole modes A_ν must vanish by the Kohn theorem [20].

The coefficient Γ_0 is unknown so far and needs to be calculated from scratch. Here we shall determine the temperature dependence of Γ_0 for large quasihomogeneous condensates for which the local density approximation is applicable, collecting the trap dependence of Γ_0 in a dimensionless prefactor which we leave undetermined. Collisions of quasiparticles with the condensate changing the particle number $|\alpha_0|^2$ in the condensate by $\Delta|\alpha_0|^2 = \pm 1$ are described by $H_3 = \int d^3x U_0[\alpha_0\psi_0(\hat{\psi}'^{+})^2\hat{\psi}' + \text{H.c.}]$. In such processes the energy changes only by a tiny amount $\Delta F = \partial F/\partial|\alpha_0|^2$ (of the order of $k_BT/\sqrt{\langle\delta N_0^2\rangle}$). However, processes described by H_3 involve three quasiparticles besides one condensate particle, and they can therefore take up, from a slightly perturbed condensate, an arbitrarily small amount ΔF of its free energy.

By the golden rule the rate $\gamma = d\langle|\alpha_0|^2\rangle/dt$ is given by

$$\gamma = -\frac{2\pi}{\hbar^2}\sum_{\nu,\mu,\kappa}\frac{1}{2}\langle|M^{(1)}_{\kappa,\nu\mu}|^2\delta(\omega_\kappa - \omega_\nu - \omega_\mu + \Delta F/\hbar) - |M^{(2)}_{\nu\mu,\kappa|}|^2\delta(\omega_\kappa - \omega_\nu - \omega_\mu - \Delta F/\hbar)\rangle \times [\bar{n}_\nu\bar{n}_\mu(\bar{n}_\kappa + 1) - (\bar{n}_\nu + 1)(\bar{n}_\mu + 1)\bar{n}_\kappa]. \tag{9}$$

The relevant matrix elements are

$$M^{(1)}_{\kappa,\nu\mu} = 2U_0\alpha_0\int d^3x\psi_0 v_\nu(u^*_\kappa u_\mu + \tfrac{1}{2}v^*_\kappa v_\mu) + (\nu \leftrightarrow \mu),$$

$$M^{(2)}_{\nu\mu,\kappa} = 2U_0\alpha_0\int d^3x\psi_0 u^*_\nu(v^*_\mu v_\kappa + \tfrac{1}{2}u^*_\mu u_\kappa) + (\nu \leftrightarrow \mu).$$

$M^{(1)}$ describes a Landau-scattering process in which one atom is scattered out of the condensate by the absorption of the two quasiparticles ν, μ out of and the emission of the new quasiparticle κ into the thermal bath. Likewise $M^{(2)}$ describes Beliaev scattering where an incoming thermal quasiparticle κ is absorbed, an atom is kicked out of the condensate, and two quasiparticles ν, μ are emitted into the thermal bath. The factor $1/2$ in (9) accounts for the indistinguishability of pairs ν, μ and μ, ν.

In the phonon part of the excitation spectrum we have $u_\lambda \simeq -v_\lambda \sim \omega_\lambda^{-1/2}$. Furthermore, in that low-energy region the statistical factor $[\cdots]$ in (9) is well approximated by $(k_BT)^2(\omega_\kappa - \omega_\nu - \omega_\mu)/\hbar^2\omega_\nu\omega_\mu\omega_\kappa$, where the frequency difference in the nominator becomes $\pm\Delta F/\hbar$ in the product with the δ functions, which express energy conservation. Anywhere else the small energy ΔF is negligible. The frequency factors in the denominator, together with similar factors in the denominator coming from the matrix elements, make the phonon contribution to the sums in (9) the dominant one, at least in large condensates, and we shall therefore concentrate on this contribution in the following. This frequency range has a natural upper cutoff at $\mu/\hbar$, where the collective phonons go over smoothly into particlelike excitations. In finite condensates also a natural lower cutoff exists at about the trap frequency $\bar{\omega}$, where the phonon wavelength becomes comparable with the size of the condensate.

We shall evaluate the sums in (9) in the usual Thomas-Fermi and local density approximation [21] for $u_\lambda, v_\lambda, \omega_\lambda$, integrating over the wave vectors of the phonons in an interval implied by the cutoffs. We obtain in this way

$$\gamma = -\frac{9A_0m^3(U_0k_BT)^2}{(2\pi)^3\hbar^8}\langle\Delta F|\alpha_0|^2\rangle \times \int\int\frac{d\omega_\nu\, d\omega_\mu}{\omega_\nu\omega_\mu(\omega_\nu + \omega_\mu)}.$$

Clearly the contribution near the lower cutoff at $\bar{\omega}$ dominates in the double frequency integral. After its evaluation, and using $U_0 = 4\pi\hbar^2a/m$ and $d_0 = \sqrt{\hbar/m\bar{\omega}}$ to introduce the s-wave scattering length a and the zero-point amplitude d_0, we obtain to leading order in $\hbar\bar{\omega}/\mu$ $d\langle|\alpha_0|^2\rangle/dt = \gamma = -2(\Gamma_0/\hbar)\langle(\partial F/\partial|\alpha_0|^2)|\alpha_0|^2\rangle$ with

$$\Gamma_0 = A_0\frac{18\ln 2}{\pi}\left(\frac{ak_BT}{d_0\hbar\bar{\omega}}\right)^2. \tag{10}$$

Here A_0 depends on the trap geometry and is of order 1 in an isotropic trap. For the experimentally realized condensates $\Gamma_0 \ll 1$ is implied by (10).

In order to give a practically useful estimate of the phase-diffusion rate let us compare the linewidth $\Delta\nu$ of the condensate with the directly measurable inverse lifetime τ_κ^{-1} of a collective mode κ with frequency of order $\bar{\omega}$. We find for $\Gamma_0 \ll 1$

$$\Delta\nu\tau_\kappa = \frac{\bar{\omega}/\omega_\kappa}{A_0A_\kappa 288\ln 2}\left[\frac{(\mu/k_BT)^2}{\langle N_0\rangle[n_0(0)a^3]^{3/2}}\right]. \tag{11}$$

For $\mu/k_BT = 10^{-1}, \langle N_0\rangle = 10^6, n_0(0)a^3 = 3\times 10^{-6}$ the factor $[\cdots]$ on the right hand side of (11) is about 2. This may explain the experimentally observed [4] robustness of the phase. Another way to put our result is to note that up to a numerical prefactor the relation $\hbar\bar{\omega}\Gamma_0/\mu \sim \Gamma_\kappa^2 = (2Q_\kappa)^{-2}$ holds. Hence, according to (7) for $\Gamma_0 \ll 1$

$$\Delta\nu = (A_\kappa^2/A_0 36\ln 2)\,(k_B T\bar{\omega}Q_\kappa^2/\mu\langle N_0\rangle)\,, \qquad (12)$$

where Q_κ is known experimentally.

In conclusion, we have provided a theory of thermally generated phase diffusion in Bose-Einstein condensates and determined a temperature dependent Schawlow-Townes-type formula for the linewidth $\Delta\nu$ of the condensate. This is also the minimum linewidth of an atom laser based on the Bose-Einstein condensate. We have also calculated the correlation time τ_c of the particle number fluctuations in the condensate which should be measurable. Last but not least we have derived a relation (8) between this correlation time τ_c and the linewidth $\Delta\nu$ which follows from the fluctuation-dissipation relation independently of any microscopic detail, but subject to the assumption of negligible squeezing in the thermal bath seen by the condensate. Removing this restriction will be the subject of a more detailed paper.

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