Integrability and Quantum Chaos in Spin Glass Shards

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We study spin glass clusters ("shards") in a random transverse magnetic field, and determine the regime where quantum chaos and random matrix level statistics emerge from the integrable limits of weak and strong fields. Relations with quantum phase transition are also discussed. [S0031-9007(98)07880-6]

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Quantum manifestations of classical chaos and integrability have been intensively investigated in the past decade [1]. It has been realized that the statistical properties of the quantum energy spectrum are strongly influenced by the underlying classical dynamics. Indeed, classical integrability generally implies the absence of level repulsion and a Poissonian statistics of energy level spacings. On the contrary, chaotic dynamics leads to level repulsion with the level spacing statistics $P(s)$ being the same as in the random matrix theory (RMT), i.e., the Wigner-Dyson (WD) distribution [2]. These two distributions of $P(s)$ also characterize, respectively, the localized and metallic phases in the Anderson model of disordered systems. At the critical point between these two phases an intermediate level spacing statistics occurs [3].

While for one-particle systems the statistical properties of spectra are well understood, the same problem in many-body systems has been addressed only recently. The first results demonstrated that many-body integrable systems are still characterized by Poisson statistics $P_P(s)$, whereas in the absence of integrals of motion the Wigner-Dyson statistics $P_{\mathrm{WD}}(s)$ has been found [4]. More recently, investigations of finite Fermi systems such as the Ce atom [5] and the ^{28}Si nucleus [6] put forward the question of the statistical description and thermalization induced by interaction. A quantum chaos criterion for emergence of RMT statistics and dynamical thermalization induced by interaction was established in Ref. [7]. Namely, a crossover from Poisson statistics to the WD distribution takes place when the coupling matrix elements become comparable to the level spacing between directly coupled states. For two-body interaction, the critical interaction strength becomes exponentially larger than the multiparticle level spacing. Indeed, the latter decreases exponentially with the number of particles, while the former decreases typically only quadratically. This result was corroborated in Ref. [8], and should apply to various physical systems such as complex nuclei, atoms, clusters, and quantum dots.

In this Letter, we develop and apply these concepts to disordered spin systems. These systems are of great experimental and theoretical interest [9]; in particular, quantum spin glasses have recently attracted a great deal of attention, and various approaches have been developed to investigate their properties [10]. Such important problems as zero temperature quantum phase transition, structure of the phase diagram, correlations, and susceptibility properties have been intensively investigated both analytically and numerically [11–16]. However, to the best of our knowledge, the level spacing statistics has not been studied in disordered spin systems. Here we present theoretical estimates for spin glass "shards" which determine the crossover from integrability to quantum chaos in a way analogous to the case of finite interacting fermionic systems. Numerical investigations of $P(s)$ statistics give a powerful test of this crossover both at high temperature T and at $T = 0$ where a quantum phase transition is believed to take place.

The spin glass shards we study are described by the Hamiltonian,

$$H = \sum_{i<j} J_{ij}\sigma_i^x\sigma_j^x + \sum_i \Gamma_i\sigma_i^z\,, \tag{1}$$

where the σ_i are the Pauli matrices for the spin i, and the first sum runs over all spin pairs. The local random magnetic field is represented by Γ_i uniformly distributed in the interval $[0,\Gamma]$. The exchange interactions J_{ij} are distributed in the same way in $[-J/\sqrt{n}, J/\sqrt{n}\,]$, where n is the total number of spins. For $\Gamma = 0$, this system is the classical Sherrington-Kirkpatrick spin glass model [9]. The $\sqrt{n}$ factor in the definition of J_{ij} ensures a proper thermodynamic limit at $n \to \infty$. In the limit $J/\Gamma \to 0$, one obtains a paramagnetic phase with all spins in the field direction. The opposite limit ($J/\Gamma \to \infty$) corresponds to a spin glass phase at low temperature [9].

Let us first discuss the properties of the model for highly excited states near the center of the energy band. At $J = 0$, the system is integrable, since there are n integrals of motion, and $P(s)$ should follow the Poisson distribution. When J/Γ increases, we expect integrability to be destroyed and therefore a crossover towards $P_{\mathrm{WD}}(s)$ statistics typical of RMT should take place. At $\Gamma = 0$, the model again becomes integrable since there are n operators (σ_i^x) commuting with the Hamiltonian; hence, $P(s) = P_P(s)$. The latter may seem surprising, especially in light of recent discussions on chaos in classical spin glass [17]. However, in spite of a possible complex thermodynamic behavior (Monte Carlo dynamics), the Hamiltonian dynamics of a classical spin glass is integrable. As a result, we expect another crossover from a WD statistics back to a Poissonian one for large J/Γ. This picture is confirmed by the numerical results for $P(s)$ displayed in Fig. 1. These $P(s)$ distributions were obtained for the states of the same symmetry class (S_2), namely, the number of spins up is always an even number (the interaction does not mix states with even and odd numbers of spins up). We will call the other symmetry class with an odd numbers of spins up S_1.

To analyze the evolution of the $P(s)$ distributions with respect to J, it is convenient to use the parameter $\eta = \int_0^{s_0} [P(s) - P_{\text{WD}}(s)]\,ds / \int_0^{s_0} [P_P(s) - P_{\text{WD}}(s)]\,ds$, where $s_0 = 0.4729\ldots$ is the intersection point of $P_P(s)$ and $P_{\text{WD}}(s)$ [7]. In this way $\eta = 1$ corresponds to the Poissonian case, and $\eta = 0$ to the WD distribution. The data of Fig. 1 show that the distributions with the same value of η are very close, even if J/Γ varies more than 10 times. It is convenient to determine a critical coupling strength J_c at which the crossover from $P_P(s)$ to $P_{\text{WD}}(s)$ takes place by the condition $\eta(J_c/\Gamma) = 0.3$. The variation of η with respect to J is shown in Fig. 2 for different n. The global behavior of η is in agreement with the above picture of transition between integrability and quantum chaos. Indeed, when J/Γ increases, η drops to zero and then starts to grow again back to one when the spin glass term in (1) dominates. The fact that a WD statistics sets in is not trivial. Indeed, in dimension $d = 1$ {the model (1) with nearest-neighbor coupling only and J_{ii+1} drawn randomly from $[0, J]$, studied in [10,11]} the statistics remains Poissonian ($\eta = 1$) at arbitrary J/Γ, as is shown in Fig. 2. The critical point $J/\Gamma = 1$ [11] does not manifest itself. Indeed, this $d = 1$ model can be mapped into a model of noninteracting fermions (see, e.g., [11]). Therefore the total energy is the sum of one-particle energies that generically leads to $P_P(s)$. As a result, RMT can never be applied to this model and dynamical (interaction induced) thermalization never takes place. Thermalization can appear only through a coupling to a thermal bath that was implicitly used in Refs. [16,18]. On the contrary, in the system (1) there are two transitions from integrability to chaos which determine two critical couplings J_{cp}, from the paramagnetic side, and J_{cs} from the spin glass side [$\eta(J_{\text{cs}}) = \eta(J_{\text{cp}}) = 0.3$; $J_{\text{cs}} > J_{\text{cp}}$].

In analogy with finite interacting fermionic systems [7], we expect that quantum chaos sets in when the coupling strength U becomes comparable to the spacing between directly coupled states Δ_c ($U \sim \Delta_c$). For small J in the middle of the spectrum, one has $U \sim J/\sqrt{n}$ and $\Delta_c \sim 16\Gamma/n^2$ since each state is coupled to $n(n-1)/2$ states in an energy band of 8Γ. This gives the quantum chaos border from the paramagnetic side:

$$J_{\text{cp}} \approx C_p \Gamma / n^{3/2}, \tag{2}$$

where C_p is some numerical constant. We note that J_{cp} is exponentially larger than the multiparticle spacing $\Delta_n \sim 2n\Gamma/2^n$. For large J, the transitions between unperturbed states at $J \gg \Gamma$ are determined by the Γ term in (1), and have typical value Γ. The number of such transitions, in a typical energy interval J, is n; hence, $\Delta_c \sim J/n$. This gives the quantum chaos border from the spin glass side:

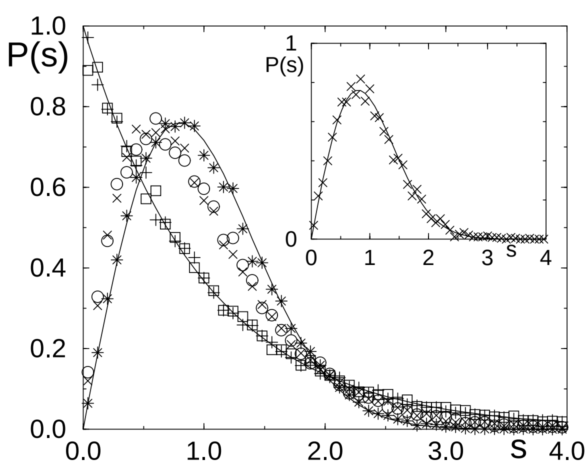

FIG. 1. Crossover from Poisson to WD statistics in the model (1) for the states in the middle of the energy band (±12.5% around the center) for $n = 12$: $J = 0$, $\eta = 0.984$ (+); $J/\Gamma = J_{\text{cp}}/\Gamma = 0.38$, $\eta = 0.3$ (×); $J/\Gamma = 0.866$, $\eta = 0.027$ (*); $J/\Gamma = J_{\text{cs}}/\Gamma = 6.15$, $\eta = 0.3$ (○); $\Gamma = 0$, $\eta = 0.99$ (□). Full curves show the Poisson and WD distributions. Total statistics (NS) is more than 3×10^4; s is in units of mean level spacing. Inset shows $P(s)$ for the first excitation from the ground state in the chaotic regime for $n = 15$, $J/\Gamma = 0.465$, $\eta_0 = 0.018$ ($NS = 3000$) (×); the full line shows $P_{\text{WD}}(s)$.

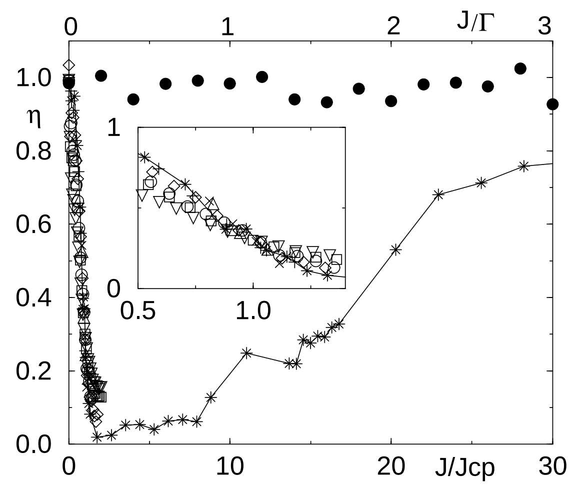

FIG. 2. Dependence of η on the rescaled coupling strength J/J_{cp} for the states in the middle of the energy band for $n = 7$ (▽), 8 (□), 9 (○), 10 (◇), 11 (+), 12 (*), 13 (×), 14 (△), and symmetry S_2; NS varies between 12 500 and 160 000. The full line for $n = 12$ shows the global behavior. Data for the $d = 1$ model (see text) are given for $n = 12$ (●) as a function of J/Γ (upper scale); $NS = 30\,000$. The inset magnifies the region near $J/J_{\text{cp}} = 1$.

$$J_{cs} \approx C_s \Gamma n\,, \tag{3}$$

where, again, C_s is some constant.

These theoretical predictions are confirmed by the numerical results presented in Figs. 2 and 3 which give $C_p \approx 16$ and $C_s \approx 0.5$. We attribute the deviation from (2) in Fig. 3 seen for small n to the fact that for these values J_{cp} is rather large; this can slightly modify the unperturbed spectrum and the estimates for Δ_c. Also, for small n the proximity of the second transition at J_{cs} can affect the actual value of J_{cp}. The data shown in the inset of Fig. 2 indicate that η depends only on the ratio J/J_{cp}; the global scaling behavior is less visible than in the fermionic model [7], apparently due to the reasons above.

The analysis at the band center corresponds to a very high energy or temperature. Therefore the properties of (1) near the ground state energy E_g should be studied separately. To do that, we investigated $P(s)$ for the first excitations from E_g within the same symmetry class. For the class S_1, the distribution $P(s)$ was computed for the first four level spacings starting from E_g by averaging over different disorder realizations. The first excitation and its η value (η_0) was analyzed separately while the other three were used to generate one cumulative $P(s)$ with an η noted by η_1 [19]. There are several physical reasons for this separation: in the presence of a gap, η_0 is related to the gap fluctuations while η_1 characterizes the quasiparticle excitations of higher energy, above the gap; in a metallic quantum dot with noninteracting electrons, η_0 reflects the chaotic character of the dot giving $\eta_0 = 0$ while the spectrum of higher excitations becomes close to $P_P(s)$ with $\eta_1 \approx 1$.

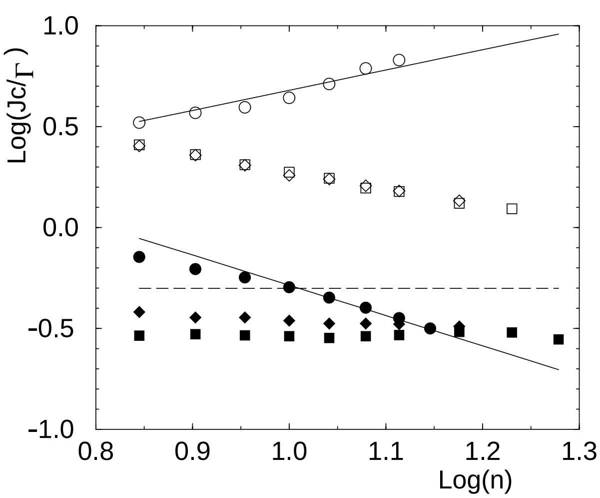

FIG. 3. Critical coupling strength J_c as a function of n: J_{cp} at band center for symmetry S_2 (●), and near the ground state (■: S_1; ◆: S_2); J_{cs} at band center for S_2 (○), and near the ground state (□: S_1; ◇: S_2). Full lines show the theory [Eqs. (2) and (3)] for the band center; the dashed line indicates the quantum critical point at $T = 0$, $J/\Gamma = 0.5$. Logarithms are decimal.

The global behavior of η_0 and η_1 as a function of $\tilde{J} = J/(J + \Gamma)$ is shown in Fig. 4. This parametrization naturally represents the two integrable limits $J = 0$ and $\Gamma = 0$ in a symmetric way. In both extreme cases ($\tilde{J} = 0; 1$), η_0 and η_1 approach the Poissonian value 1. In between there is a pronounced minimum with $\eta_0 \approx 0$ and $\eta_1 \approx 0.1$. When the number of spins in the shard increases, the values of η to the right from the minimum go up markedly, whereas the minimal η value itself changes only weakly. These η data suggest the existence of a crossing point around $J = J_q \approx 0.5\Gamma$. For $J < J_q$, the difference between curves for different n is small, whereas it is much bigger for $J > J_q$. The data in Fig. 3 show the variation of J_{cp}, J_{cs} (for which $\eta_1 = 0.3$) with n; the behavior of J_{cs} is consistent with $J_{cs} \to J_q$ for $n \to \infty$ for both symmetry classes. Also, according to these data $J_{cp} \propto n^{-\alpha}$ with $0 \leq \alpha \leq 0.2$. This value of α is smaller than the one expected from the quantum chaos criterion, used to derive (2). Indeed, for small J near the ground state $\Delta_c \sim \Gamma/n$ whereas the coupling is still $J/\sqrt{n}$, and therefore one expects $\alpha = 0.5$. For large n, this would imply that η drops quickly to zero with J and then increases again after the crossing point J_q. In such a scenario, in the thermodynamic limit ($n \to \infty$) we expect $J_{cp} \to 0$ and $J_{cs} \to J_q$, so that η changes from zero ($J < J_q$) to one ($J > J_q$) with some intermediate statistics at the critical point J_q. The determination of the η value at that point for $n \to \infty$ requires larger system sizes. We expect that this critical point corresponds to the quantum phase transition between paramagnetic and spin glass phases discussed in Refs. [10,12,15] for a case when all $\Gamma_i = \Gamma$. The above scenario is similar to the situation for the Anderson transition in $d = 3$ [3]: as for the Anderson insulator, the lack of space ergodicity in the spin glass phase implies $\eta = 1$. In this phase

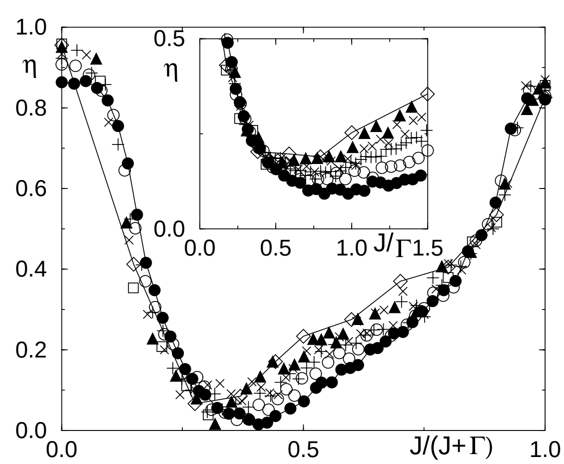

FIG. 4. Dependence of η_0 on $\tilde{J} = J/(J + \Gamma)$: $n = 7$ (●), $n = 9$ (○), $n = 11$ (+), $n = 13$ (×), $n = 15$ (▲), $n = 17$ (◇), $n = 19$ (□); $2000 \leq NS \leq 30\,000$. Full curves connect data for $n = 7, 17$. Inset shows η_1 in the region near $J/\Gamma = J_q/\Gamma \approx 0.5$ in more detail; $6000 \leq NS \leq 90\,000$. Data are for S_1 symmetry.

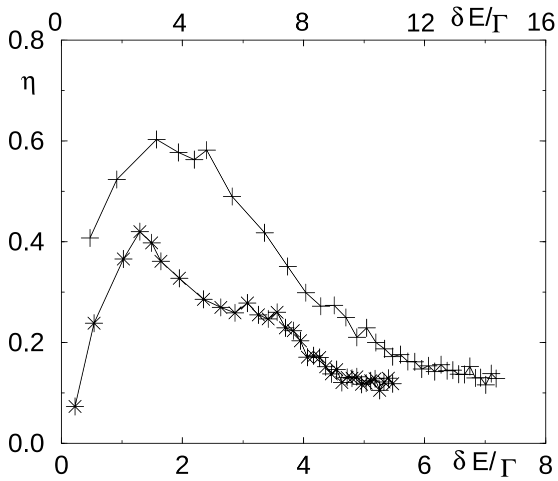

FIG. 5. Dependence of η on $\delta E/\Gamma$ for $n = 12$ (1000 realizations of disorder): $J/\Gamma = 0.52$ (*) (lower scale); $J/\Gamma = 3.46$ (+) (upper scale). Data are for S_1 symmetry.

$J > J_q$ the eigenstates are expected to be localized in different parts of the phase space with small overlap between them. This would lead to uncorrelated levels (Poisson distributed) as in integrable systems. Even if in this regime the levels can be rather sensitive to a small parameter variation [17], the eigenstates are not ergodic and the usual properties of quantum chaos [1,2] are absent. The situation in the paramagnetic phase $J < J_q$ may be more complicated. Indeed, the criterion $U \sim \Delta_c$ which gives $\alpha = 0.5$ assumes that the couplings remain small and do not strongly modify the unperturbed Δ_c. But near E_g the typical mean field acting on a spin is of the order J and for $J \approx \Gamma/\sqrt{n}$ is much larger than the local fields $\Gamma_i \sim \Gamma/n$. The strong mean field near E_g can change the effective Δ_c. Such a fact was seen for the Ce atom [5]. This may lead to another scenario in which in the thermodynamic limit $J_{\rm cp}$ tends to some nonzero value smaller than J_q. Such a behavior would be in agreement with the small variation of $J_{\rm cp}$ with n in Fig. 3.

The discussions above dealt with the two extreme cases of energy values. The variation of η between these two limits is shown in Fig. 5 near the critical point and in the spin glass phase as a function of the excitation energy $\delta E = E - E_g$. Both cases show an unusual behavior when η initially grows with energy and starts to decrease only later. This tendency is more pronounced near the critical point. A possible reason for this behavior is that the relative influence of the mean field becomes weaker as $\delta E/\Gamma$ increases. It is also possible that the situation is similar to the case of a chaotic quantum dot with noninteracting electrons discussed above: the ground state is chaotic but the interaction between the first quasiparticle excitations is weak and so initially η grows with δE, approaching the Poissonian value. As a result, it is only at higher energy when the interaction between quasiparticles becomes stronger that η starts to decrease towards the WD value. Larger system sizes are required to clarify this issue.

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[1] *Chaos and Quantum Physics*, Les Houches Lecture Series Vol. 52, edited by M.-J. Giannoni, A. Voros, and J. Zinn-Justin (North-Holland, Amsterdam, 1991).
[2] O. Bohigas, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984); O. Bohigas, in *Chaos and Quantum Physics* (Ref. [1]).
[3] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, Phys. Rev. B **47**, 11 487 (1993); D. Braun, G. Montambaux, and M. Pascaud, Phys. Rev. Lett. **81**, 1062 (1998).
[4] G. Montambaux, D. Poilblanc, J. Bellissard, and C. Sire, Phys. Rev. Lett. **70**, 497 (1993); D. Poilblanc, T. Ziman, J. Bellissard, F. Mila, and G. Montambaux, Europhys. Lett. **22**, 537 (1993).
[5] V. V. Flambaum, A. A. Gribakina, G. F. Gribakin, and M. G. Kozlov, Phys. Rev. A **50**, 267 (1994); V. V. Flambaum, A. A. Gribakina, G. F. Gribakin, and I. V. Ponomarev, Phys. Rev. E **57**, 4933 (1998).
[6] V. Zelevinsky, B. A. Brown, N. Frazier, and M. Horoi, Phys. Rep. **276**, 85 (1996).
[7] P. Jacquod and D. L. Shepelyansky, Phys. Rev. Lett. **79**, 1837 (1997).
[8] B. Georgeot and D. L. Shepelyansky, Phys. Rev. Lett. **79**, 4365 (1997); D. Weinmann, J.-L. Pichard, and Y. Imry, J. Phys. I (France) **7**, 1559 (1997); A. D. Mirlin and Y. V. Fyodorov, Phys. Rev. B **56**, 13 393 (1997); R. Berkovits and Y. Avishai, Phys. Rev. Lett. **80**, 568 (1998).
[9] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986); M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[10] *Spin Glasses and Random Fields*, edited by A. P. Young (World Scientific, Singapore, 1997).
[11] D. S. Fisher, Phys. Rev. Lett. **69**, 534 (1992); Phys. Rev. B **51**, 6411 (1995).
[12] J. Miller and D. A. Huse, Phys. Rev. Lett. **70**, 3147 (1993).
[13] M. Guo, R. N. Bhatt, and D. A. Huse, Phys. Rev. Lett. **72**, 4137 (1994); Phys. Rev. B **54**, 3336 (1996).
[14] H. Rieger and A. P. Young, Phys. Rev. Lett. **72**, 4141 (1994); Phys. Rev. B **54**, 3328 (1996).
[15] D. R. Grempel and M. J. Rozenberg, Phys. Rev. Lett. **80**, 389 (1998); **81**, 2550 (1998).
[16] S. Sachdev, cond-mat/9705266.
[17] A. J. Bray and M. A. Moore, Phys. Rev. Lett. **58**, 57 (1987); M. Alava and H. Rieger, cond-mat/9804136.
[18] S. Sachdev and A. P. Young, Phys. Rev. Lett. **78**, 2220 (1997).
[19] In the S_2 symmetry class, $\eta_0 \approx 0.4$ at $J = 0$; this complicates the analysis of η_0, but the results for η_1 in Fig. 3 are similar to those of S_1.