Quantum Metamorphosis of a Conformal Transformation in D3-Brane Yang-Mills Theory

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We show how the linear special conformal transformation in four-dimensional $N = 4$ super-Yang-Mills theory is metamorphosed into the nonlinear and field-dependent transformation for the collective coordinates of Dirichlet 3-branes, which agrees with the transformation law for the space-time coordinates in the anti–de Sitter (AdS) space-time. Our result provides a new and strong support for the conjectured relation between $AdS_5 \times S^5$ supergravity and super-Yang-Mills theory (SYM). Furthermore, our work sheds elucidating light on the nature of the AdS/SYM correspondence. [S0031-9007(98)07828-4]

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One of the most interesting recent outcomes from studies of various duality relations in superstring theories is the correspondence between the large N super-Yang-Mills (SYM) theories describing the low-energy dynamics of Dirichlet branes and supergravities in the background of anti–de Sitter (AdS) space-times. Based on some earlier results [1], the precise criteria for the validity of such correspondence are discussed in [2] and the concrete formulation of the relations between correlation functions on both sides are proposed in [3,4]. From the viewpoint of string theory, the correspondence may be regarded as a special version of old s-t duality which relates open strings in s channel and closed strings in t channel. In connection with this view, it has been emphasized that, at the heart of this remarkable relation, there is an underlying duality between the short and large distances on each side, the "space-time uncertainty relation" in the language of [5,6] or "ultraviolet-infrared relation" using the terminology of [7].

As far as we can see, however, the basis for the correspondence is yet phenomenological in nature, and no logical or "intrinsic" explanation *within* the framework of super-Yang-Mills theory has been known, except for some analogy with lower dimensional examples such as those between the three-dimensional Chern-Simon theories in the bulk and the corresponding two-dimensional conformal-field theories (CFTs) at the boundary. Very recently, some works [8–11] trying to fill this gap appeared. In the present Letter, we provide another approach aiming toward such a goal.

One of the handles in pursuing such an explanation is the (super-) conformal symmetry on both sides. As is well known, the four-dimensional conformal group of 4D Yang-Mills theory is isomorphic to the isometry group of the five-dimensional anti–de Sitter space-time, AdS_5. In the coordinate frame most appropriate for making comparison with the standard formulation of Yang-Mills theory, the metric on the supergravity side, $AdS_5 \times S^5$, is

$$ds^2 = \alpha'\left(\frac{R^2}{U^2}(dU^2 + U^2 d\Omega_5^2) + \frac{U^2}{R^2}dx_4^2\right), \tag{1}$$

where (and throughout the present paper) we use the same conventions as Ref. [2]. Thus U is the radial coordinate measured in the energy unit, $U = r/\alpha'$. The throat radius of the AdS space-time in the dimesionless unit is $R = (2g_{\rm YM}^2 N)^{1/4}$, and the Yang-Mills coupling is related to the standard string coupling $g_s = e^{\phi}$ by $g_{\rm YM}^2 = 2\pi g_s$. The special conformal transformation for the longitudinal four-dimensional coordinates x^a $(a = 0, 1, 2, 3)$ and the radial coordinate U as a part of the isometry of this metric are

$$\delta_K x^a = -2\epsilon \cdot x x^a + \epsilon^a x^2 + \epsilon^a \frac{R^4}{U^2}, \tag{2}$$

$$\delta_K U = 2\epsilon \cdot x U. \tag{3}$$

In the usual interpretation of the Yang-Mills theory as the boundary field theory corresponding to supergravity in the bulk, the ordinary transformation law on the Yang-Mills side is identified with these transformations in the limit $U \to \infty$ of (2) and (3). This is certainly a consistent interpretation.

On the other hand, from the viewpoint of effective world-volume theory for D3-brane, the Higgs fields of the $N = 4$ super-Yang-Mills theory must play the role of the collective coordinates which are transversal to the D3-brane, and hence correspond to the directions described by the radial coordinate U together with the angle coordinates describing S^5 in the bulk theory. The above metric should therefore be detected in the dynamics of these coordinates representing a probe D3-brane in the presence of the background corresponding to the heavy source described by a large number of coincident D3-branes at rest at the origin. From this point of view, the Yang-Mills theory as a whole *cannot* be regarded as living on the boundary (or anywhere) of the AdS space, since the above interpretation crucially depends on the choice of

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D-brane configuration as the background of the Yang-Mills theory. Clearly, the question "Where are the branes?" can meaningfully be asked only after a choice is made for the background in Yang-Mills theory.

One way of approaching this picture is of course to try to compute the effective action for the probe D3-brane from the Yang-Mills side [8], just as we do for D-particles [12,13] in matrix theory. In the following, we take a different approach. Namely, we try to derive the transformation law in the bulk theory directly from Yang-Mills theory. In other words, we shall clarify how such a field-dependent transformation law can emerge from the ordinary linear transformation law.

The classical Yang-Mills action in our convention is

$$S_{d3} = -\int d^4x \frac{1}{4g_{\mathrm{YM}}^2} \times \mathrm{Tr}\left(F_{ab}F^{ab} + \frac{2}{(2\pi\alpha')^2} D_a X^\mu D^a X_\mu + \frac{1}{(2\pi\alpha')^4}[X^\mu, X^\nu]^2\right) + \dots, \tag{4}$$

where we suppressed the fermionic part. Here the space-time indices μ, ν for the Higgs fields X^μ run through the transverse directions from 4 to 9 and the world-volume coordinates are identified with the space-time coordinates in the longitudinal directions $a = 0, \dots, 3$ assuming the static gauge for the parametrization in flat world volume. The action is invariant under the ordinary special conformal transformation generated by

$$\delta_K A_a(x) = (\delta_K x^b)\partial_b A_a(x) - 2\epsilon \cdot x A_a(x) + 2x_a \epsilon \cdot A(x) - 2\epsilon_a x \cdot A(x), \tag{5}$$

$$\delta_K X^\mu(x) = (\delta_K x^b)\partial_b X^\mu(x) - 2\epsilon \cdot x X^\mu(x). \tag{6}$$

The effective dynamics of D3-branes is described by the diagonal matrix elements of the Higgs fields. If the distance between the source and the probe is first assumed to be large, the energy scale of the off-diagonal part is large or the length scale in the world volume is small, and it is appropriate to integrate over the off-diagonal part, keeping fixed the low-energy (or large-distance) dynamics of the diagonal part. In order to carry this out, we have to fix the gauge for the off-diagonal part. The most convenient is the usual background-field gauge, assuming the diagonal part B of the fields as the background fields. Namely, the gauge function is

$$F = \partial_a A^a - i\frac{1}{(2\pi\alpha')^2}[B_\mu, Y^\mu],$$

where we have denoted the off-diagonal part of the Higgs fields by Y^μ, i.e., $X^\mu = B^\mu + Y^\mu$.

Let us now consider the effect of the special conformal transformation on the gauge function F. We find

$$\delta_K F = (\delta_K x^a)\partial_a F - 2\epsilon \cdot x F + 4\epsilon \cdot A. \tag{7}$$

Thus the gauge condition cannot be invariant under the special conformal transformation and, hence, we have to perform a field dependent gauge transformation to recover the original gauge condition. The required gauge parameter is

$$\Lambda = 4\triangle_B^{-1}\epsilon \cdot A, \tag{8}$$

where $\triangle_B$ is defined by

$$\triangle_B \Lambda = -D^a \partial_a \Lambda + \frac{1}{(2\pi\alpha')^2}[B_a + Y_a, [B_a, \Lambda]],$$

which of course is the kinetic operator for the ghost action $\int d^4x\, \mathrm{Tr}(\overline{C}\triangle_B C)$. Thus the special conformal transformation for the Higgs fields is modified as

$$\tilde{\delta}_K X = \delta_K X + 4i[\triangle_B^{-1}\epsilon \cdot A, X]. \tag{9}$$

A similar modification of the special conformal transformation was essentially noticed long ago in [14], in the case of Feynman gauge. It is straightforward to check that the measure is invariant under this transformation using the BRS formalism following Ref. [14]. In the BRS formalism, the modified term is supplied from the field-dependent BRS transformation whose Jacobian compensates the violation of the special conformal invariance of the gauge fixing term $\int d^4x \frac{1}{2g_s}\,\mathrm{Tr}\, F^2$. Integration over the ghost fields gives the same final form for the modified term.

Because of this modification, the conformal Ward identity for the effective action Γ for the diagonal parts, which is the sum over the diagrams one–particle irreducible (1PI) with respect to the diagonal Higgs fields B, takes the form

$$\int d^4x\, (\delta_K B + 4i\langle[\triangle_B^{-1}\epsilon \cdot A, X]_{\mathrm{diagonal}}\rangle)\frac{\delta\Gamma}{\delta B} = 0, \tag{10}$$

where the subscript "diagonal" for the commutator indicates that only the diagonal part is taken and the bracket $\langle\cdot\rangle$ indicates the expectation value with respect to the path integral over the off-diagonal part.

We can now evaluate the expectation value of the additional term. For small transverse velocities, the lowest nontrivial contribution comes from the diagram with one vertex

$$-\frac{2}{g_{\mathrm{YM}}^2(2\pi\alpha')^2}\int d^4x\, \mathrm{Tr}(\partial_a B^\mu[Y_\mu, A_a])$$

inserted, which mixes the gauge and the Higgs fields. Since this vertex is already of first order in velocity, we can use the static approximation for the rest. Then the result for the probe D3-brane in the presence of N coincident source D3-branes at the origin is

$$4i\langle[\triangle_B^{-1}\epsilon \cdot A, X]_{\mathrm{diagonal}}\rangle = 16ig_{\mathrm{YM}}^2 N\int \frac{d^4k}{(2\pi)^4} \times \frac{1}{(k^2 + M^2 - i\epsilon)^3}\epsilon\partial B = \frac{g_{\mathrm{YM}}^2 N}{2\pi^2 M^2}\epsilon\partial B,$$

where $M^2 = (\frac{r}{2\pi\alpha'})^2$. Note that the background Higgs fields take the following form ($r^2 = r^\mu r_\mu$):

$$B^\mu = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & r^\mu \end{pmatrix},$$

where the only nonzero entry is the last $(N + 1, N + 1)$ matrix element corresponding to the probe D3-brane and the N zero diagonal elements represent the source D3-branes which can be assumed to be at rest in the large N limit. Note also that the modification of the conformal transformation of the source D3-branes can be neglected compared with that of the probe in the large N limit.

The above result, *including the numerical coefficient*, precisely gives the last term in the transformation law for the world-volume (i.e., longitudinal) coordinates of the anti–de Sitter space-time (2) since $R^4/U^2 = (2\pi\alpha')^2 g_{\rm YM}^2 N/2\pi^2 r^2$. The transformation law of the diagonal Higgs fields now takes the AdS form (3) corresponding to the radial coordinate U. Remember here that the transformation of the world-volume coordinate and the scaling of the fields are oppositely related to each other. We have thus succeeded in deriving the transformation law in the bulk starting from the "boundary" conformal field theory. We would like, however, to remind the reader that our interpretation is somewhat different as already emphasized before. Also, our derivation shows that in general the transformation law is subject to higher order corrections both in velocities and in the Yang-Mills coupling constant. Furthermore, the corrections can in principle be computed using the conformal Ward identities for arbitrary backgrounds. For general backgrounds, however, the modifed transformation rule cannot be interpreted in terms of simple space-time picture based on classical geometrical language. This further suggests that some kind of collective field theory which describes the dynamics of the diagonal part after eliminating the off-diagonal part might be a convenient tool for establishing the AdS/SYM correspondence in a more general way. A discussion concerning the relevance of collective field theory is given in [15].

As shown in [2], the modified transformation law is very powerful in determining the effective action for the D3-brane on the AdS background in the low-velocity approximation. Combined with a few assumptions, in particular, a supersymmetric nonrenormalization theorem, the form of the action in the leading approximation in the velocity expansion is uniquely determined to all *classical* orders in the string coupling, coinciding with the familiar Born-Infeld action. Thus our result implies that the probe D3-brane described by the Yang-Mills theory must detect the anti–de Sitter space-time described by the metric (1), since the metric is uniquely characterized by the transformation law (2) and (3).

At this juncture, let us comment once more on the interpretation of the AdS/SYM correspondence. Our result indicates that the super-Yang-Mills theory in this correspondence can indeed be interpreted as the effective theory of D3-branes applicable to arbitrary background configurations of D3-branes. The AdS transformation law emerges upon integrating over the off-diagonal degrees of freedom, which, in the picture of s-t duality emphasized in the beginning of the present paper, represent the dynamics of closed strings in the t channel using the dual s-channel language. If there are a large number, N_1, of probe D3-branes at a sufficiently large radial distance U from the source consisting of $N_2(\gg N_1 \gg 1)$ coincident D3-branes, the system of the probe D-branes is treated as $\mathrm{U}(N_1)$ Yang-Mills theory, and the conformal transformation reduces to the usual linear one. It seems that the boundary conformal field theory is identified with either the probe or the source Yang-Mills theory with reduced gauge group $\mathrm{U}(N_1)$ or $\mathrm{U}(N_2)$, respectively. This suggests that the AdS/SYM correspondence may possibly be proven by establishing that the gauge-broken part, $\mathrm{U}(N_1 + N_2)/\mathrm{U}(N_1) \times \mathrm{U}(N_2)$, of the whole $\mathrm{U}(N_1 + N_2)$ Yang-Mills theory describes supergravity in the low-energy and the large N_1, N_2 limit. As is more or less evident from our discussions, the low-energy (or long-distance) effects on the supergravity side are in turn related to the high-energy or short-distance effects on the Yang-Mills side, reflecting the space-time uncertainty or "UV/IR" relation. This makes the above scenario for a possible derivation of the AdS/SYM correspondence conceptually feasible.

In Ref. [5], two of the present authors argued that the conformal symmetry can be extended to the case of D-particles and may be useful for discussing the dynamics of D-particles in almost the same sense as for D3-branes. The present derivation of the bulk conformal transformation for D3-branes stemmed from our investigation along this line. Actually, however, the extension of the present formalism to other "dilatonic" branes including D-particles requires some more intricate treatments and will be discussed elsewhere [16]. It is also interesting to see whether the present method can be extended to other nondilatonic examples [2] of the AdS/CFT correspondence.

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[1] I. Klebanov, Nucl. Phys. **B496**, 231 (1997); S. Gubser, I. Klebanov, and A. Tseytlin, Nucl. Phys. **B499**, 217 (1997); S. Gubser, I. Klebanov, and A. Peet, Phys. Rev. D **54**, 3916 (1996).
[2] J. Maldacena, hep-th/9711200.
[3] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, hep-th/9802109.
[4] E. Witten, hep-th/9802150.
[5] A. Jevicki and T. Yoneya, hep-th/9805069.
[6] M. Li and T. Yoneya, hep-th/9806240.
[7] L. Susskind and E. Witten, hep-th/9805114.
[8] M. R. Douglas and W. Taylor IV, hep-th/9807225.
[9] M. Berkooz, hep-th/9807230.
[10] T. Banks, M. R. Douglas, G. Horowitz, and E. Martinec, hep-th/98708016.
[11] V. Balasubramanian, P. Kraus, A. Lawrence, and S. P. Trivedi, hep-th/9808017.
[12] K. Becker, M. Becker, J. Polchinski, and A. Tseytlin, Phys. Rev. D **56**, 3174 (1997).
[13] Y. Okawa and T. Yoneya, hep-th/9806108.
[14] E. S. Fradkin and M. Palchik, Phys. Lett. **147B**, 86 (1984); see also E. S. Fradkin and M. Palchik, Phys. Rep. **300**, 1 (1998).
[15] A. Jevicki, in *Proceedings of the Fourth Workshop on Quantum Chromodynamics, Paris, 1998* (World Scientific, Singapore, 1998).
[16] A. Jevicki, Y. Kazama, and T. Yoneya, hep-th/9810146.