Comment on "Theorem for Nonrotating Singularity-Free Universes"

In Ref. [1], Raychaudhuri has proved an interesting result applicable to nonrotating *open* cosmological models giving an important insight into the question of the existence of singularity-free cosmological models. Essentially, the result states that in any nonrotating singularity-free universe satisfying the strong energy condition (SEC) the *spacetime average* of the energy density (and similar quantities) must vanish. This is true for the explicit models known hitherto (see [2,3] and references therein). However, a closer look at Raychaudhuri's result reveals that this property is not exclusive of singularity-free models, but rather a remarkable property of *most* nonrotating open cosmologies. As an illustrative example, the computation of the averages for the simplest standard Friedmann-Lemaître-Robertson-Walker (FLRW) models will be given below and will show to vanish in general.

The spacetime average of any f over a region $\mathcal{D}$ is

$$\langle f\rangle_{\mathcal{D}} \equiv \frac{\int_{\mathcal{D}} f\eta}{\int_{\mathcal{D}} \eta} \equiv \frac{\int_{\mathcal{D}} f\eta}{V_{\mathcal{D}}}, \tag{1}$$

where η is the volume element 4-form so that $V_{\mathcal{D}}$ is the 4-volume of $\mathcal{D}$. The spacetime average $\langle f\rangle$ of f is the limit of (1) when $\mathcal{D}$ is the whole spacetime. For the nonsingular perfect-fluid solutions of [2], $\langle\rho\rangle$, $\langle p\rangle$, and $\langle\theta\rangle$ vanish, where ρ, p, and θ are the fluid energy density, pressure, and expansion, respectively. This was generalized in [1] to nonrotating open singularity-free models satisfying SEC.

Is this a genuine property of nonsingular models? To answer this, let us compute the spacetime averages in the flat FLRW spacetimes

$$ds^2 = -dt^2 + a^2(t)\,(dx^2 + dy^2 + dz^2).$$

By assuming the simplest equation of state $p = \gamma\rho(-1 < \gamma < 1)$, $a(t)$ and $\rho(t)$ are given by (units $8\pi G = c = 1$)

$$a(t) = \text{const} \times t^{2/3(1+\gamma)}, \qquad \rho(t) = \frac{4}{3(1+\gamma)^2}\frac{1}{t^2} \tag{2}$$

with the big bang placed at $t \to 0$. Use as region $\mathcal{D}$ any 4-"rectangle" limited by values $t_0 > t_1 > 0$. A straightforward calculation using (1) and (2) provides

$$\langle\rho\rangle_{\mathcal{D}} = \frac{4(3+\gamma)}{3(1+\gamma)^2(1-\gamma)} \times \left(\frac{t_1^{(1-\gamma)/(1+\gamma)} - t_0^{(1-\gamma)/(1+\gamma)}}{t_1^{(3+\gamma)/(1+\gamma)} - t_0^{(3+\gamma)/(1+\gamma)}}\right)$$

from which we immediately get

$$\langle\rho\rangle = \lim_{t\to\infty} \frac{4(3+\gamma)}{3(1+\gamma)^2(1-\gamma)}\, t^{-2} = 0$$

so that the spacetime average of ρ and p vanish for these simple models. This holds for any open spatially homogeneous model. Therefore, the vanishing of the spacetime matter averages is a property *shared* by all models. In the singular ones, matter quantities blow up at the singularity, but this is just the *defining* difference with the nonsingular cosmologies. Thus, the word "empty" used in [1] to qualify the nonsingular models seems unfortunate, for it could be applied to the FLRW models, which are nonempty from any possible viewpoint.

In conclusion, let us stress that pure *spatial* averages (at any t) of the matter quantities also vanish in the nonsingular models [2], while they do not vanish for spatially homogeneous models. Intuitively, the vanishing of spatial averages is fundamental for the absence of singularities, and perhaps the original ideas in [1] may be improved as follows. In most regular cases, Eq. (2) in [1] implies that $\Sigma{:}x^0 = \text{const}$ are Cauchy hypersurfaces. Then, application of Raychaudhuri's equation [1,4] leads to a singularity in the finite past if the expansion of the geodesic congruence orthogonal to Σ is bounded from below by a positive constant at Σ [3,5]. Thus, nonsingular everywhere expanding models need the vanishing of the expansion at spatial infinity, implying in many cases that *spatial* averages of the scalars appearing in Raychaudhuri's equation vanish. All in all, we put forward the following.

Conjecture.—In any singularity-free nonrotating everywhere expanding globally hyperbolic model, if SEC holds, the spatial averages of the matter scalars vanish.

In yet other words, if the *spatial* averages do not vanish, then spacetime must be singular to the past. Thus, it seems that a possible satisfactory distinguishing factor between singular and nonsingular models could be the vanishing of the spatial average of the energy density.

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[1] A. K. Raychaudhuri, Phys. Rev. Lett. **80**, 654 (1998).
[2] J. M. M. Senovilla, Phys. Rev. Lett. **64**, 2219 (1990); E. Ruiz and J. M. M. Senovilla, Phys. Rev. D **45**, 1995 (1992); J. M. M. Senovilla, Phys. Rev. D **53**, 1799 (1996).
[3] J. M. M. Senovilla, Gen. Relativ. Gravit. **30**, 701 (1998).
[4] A. K. Raychaudhuri, Phys. Rev. **98**, 1123 (1955).
[5] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).

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