Nonlocal Bottleneck Effect in Two-Dimensional Turbulence

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The bottleneck pileup in the energy spectrum is investigated for several two-dimensional (2D) turbulence systems by numerical simulation using high-order diffusion terms to amplify the effect, which is weak for normal diffusion. For 2D magnetohydrodynamic (MHD) turbulence, 2D electron MHD (EMHD) turbulence, and 2D thermal convection, which all exhibit direct energy cascades, a nonlocal behavior is found resulting in a logarithmic enhancement of the spectrum. [S0031-9007(98)07802-8]

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The local enhancement of the energy spectrum in front of the dissipation range, which is now generally called the bottleneck effect, is a well-established phenomenon. It has been observed in numerous experiments [1,2] and numerical simulations [3–5], and has been discussed theoretically [6] pointing out the physical mechanism. Even a quantitative formula was derived assuming a Batchelor fit for the second-order structure function [7,8]. It is interesting to note that the magnitude of the bottleneck effect, however, seems to depend on the character of the turbulent eddies. In simulations of supersonic turbulence [9] the irrotational compressible part of the velocity field exhibits a considerably weaker spectral enhancement than the solenoidal part. For higher-order dissipation terms $\nu\nabla^2 \rightarrow -\nu_n(-\nabla^2)^n$, as often used in turbulence simulations to maximize the inertial range, the amplitude of the bottleneck effect increases, such that for $n \gg 1$ it seems to affect also the low-k inertial range behavior [10], though these results are probably not asymptotic.

Contrary to the attention the bottleneck effect attracted in three-dimensional (3D) turbulence, it has to our knowledge not yet been discussed in 2D turbulent systems. It is true that for the enstrophy cascade in 2D Euler turbulence no such effect exists, which is attributed to the negative sign of the eddy viscosity [11], making the energy spectrum slightly steeper than the corresponding Kolmogorov law, $E_k \sim k^{-3}(\ln k/k_0)^{-1/3}$. But for 2D systems dominated by a direct energy cascade there is no *a priori* argument, why the same mechanism leading to the bottleneck effect in 3D should not also be active in 2D. The effect seems, however, to be much weaker, since numerical simulations of such turbulent systems, in particular in 2D magnetohydrodynamics (MHD) [12–14] and 2D electron magnetohydrodynamics (EMHD) [15,16] have found energy spectra exhibiting almost perfect power laws down to the dissipative falloff with no visible bottleneck pileup.

One could argue that the difference is only due to the geometry of the triad interactions, now restricted to one plane, and that applying the analysis of Ref. [8] to a 2D system might lead to a less pronounced effect than in 3D. The only change in the algebra is to replace the integral in the expression for $E_k \propto \int_0^\infty kr \sin krD(r)\,dr$, see [8], by $\int_0^\infty krJ_0(kr)D(r)\,dr$, where $D(r) = \langle[v_r(\mathbf{x}+\mathbf{r}) - v_r(\mathbf{x})]^2\rangle$ is the second-order longitudinal structure function. Assuming again a Batchelor fit for $D(r)$, a straightforward evaluation shows that the bump on the spectrum would be of similar magnitude and width as in the 3D case, contrary to the much weaker effect revealed in 2D simulations, which invalidates this assumption. It is therefore necessary to investigate the character of the transition from the inertial to the dissipation range in 2D turbulence more in detail.

In this Letter we present results of a series of simulation runs for the turbulence systems mentioned above, 2D MHD, 2D EMHD, and also 2D thermal convection [19], using high n in order to amplify the inherently weak 2D bottleneck effect and going to higher spatial resolution than done previously. All three systems are two-field models which, though formally of similar structure, exhibit rather different turbulence properties. Here the main interest is, however, not in the physics described by these models, for which we refer to the original papers. We first consider EMHD turbulence, which is most closely related to (3D) Navier-Stokes turbulence. The 2D EMHD equations are [15]

$$(\partial_t + \mathbf{v}_e \cdot \nabla)(\psi - d_e^2 j) = -\eta_n(-\nabla^2)^n\psi\,, \quad (1)$$

$$(\partial_t + \mathbf{v}_e \cdot \nabla)(\phi - d_e^2\omega) + \mathbf{B}\cdot\nabla j = -\eta_n(-\nabla^2)^n\omega\,, \quad (2)$$

where the flux function ψ describes the magnetic field in the plane, $\mathbf{B} = \mathbf{e}_z \times \nabla\psi, j = \nabla^2\psi$, and the stream function ϕ describes the electron flow in the plane, $\mathbf{v}_e = \mathbf{e}_z \times \nabla\phi, \omega = \nabla^2\phi$. $\mathbf{v}_e$ is proportional to the current density in the plane, such that ϕ gives the out-of-plane field fluctuation, $\phi = \delta B_z$. The equations are written in nondimensional form and $d_e = c/\omega_{pe}L$ is the normalized collisionless electron skin depth; for details, see [15]. The equations are solved on a periodic box of linear size 2π using a standard pseudospectral method with dealiasing according to the 2/3 rule. The dissipation terms are integrated exactly. As in [15] we consider turbulence decaying from a random initial state. It has been shown in [15] that for large wave numbers $kd_e > 1$, 2D EMHD turbulence exhibits a Kolmogorov energy spectrum $E_k \sim \epsilon^{2/3}k^{-5/3}$.

Figure 1 gives the energy spectrum $\overline{E}_k = \langle E_k \epsilon^{-2/3} \rangle_t$ averaged over about one energy decay time. (Since at high n the dissipation length is essentially independent of ϵ, the average can be performed at constant k [17].) Shown are three cases with $d_e = 0.3, N^2 = 2048^2$, $k_{\max} = 682$, chosen as in [15], (a) $n = 3, \eta_3 = 6 \times 10^{-11}$, (b) $n = 8, \eta_8 = 10^{-38}$, (c) $n = 20, \eta_{20} = 10^{-96}$. The modal energy is defined by $E_k = \sum_{\text{angle}}(k^2|\psi_k|^2 + |\phi_k|^2)(1 + d_e^2k^2)$. While no bottleneck effect is visible for $n = 3$, in agreement with the spectrum shown in [15], there is a clear spectral enhancement for $n = 8$ of 20% and for $n = 20$ of about a factor of 2. For comparison with the corresponding 3D behavior several 3D EMHD simulation runs have been performed for similar parameter values, though at lower Reynolds numbers. 3D EMHD follows the equation

$$\partial_t(\mathbf{B} - d_e^2\nabla^2\mathbf{B}) - \nabla \times [\mathbf{v}_e \times (\mathbf{B} - d_e^2\nabla^2\mathbf{B})] = -\eta_n(-\nabla^2)^n\mathbf{B}\,, \quad (3)$$

where $\mathbf{v}_e = -\nabla \times \mathbf{B}$. In Fig. 2 we plot the compensated energy spectrum from three simulation runs of decaying 3D turbulence with $N^3 = 256^3, k_{\max} = 85$, for $d_e = 1$ and $n = 3, 8, 20$, which show bottleneck enhancement factors of $2.5, 4, 10$, respectively. Hence the bottleneck effect is indeed quantitatively much weaker in 2D than in 3D. (Note that for $kd_e \gg 1$ 3D EMHD formally reduces to the Navier-Stokes equation in the vorticity form $\nabla^2\mathbf{B} = \nabla \times \mathbf{v}_e \rightarrow \nabla \times \mathbf{v}$. In fact the $n = 8$ spectrum in Fig. 2 is practically identical with that observed for the corresponding Navier-Stokes case [17].)

The general understanding of the bottleneck seems to be that the spectral enhancement, while depending on n, is independent of the extent of the inertial range. Increasing the Reynolds number should only shift the bump to larger k, but not increase its amplitude; see, e.g., [8]. We will now show that such behavior is in general not true for 2D turbulence. Figure 3 gives the energy spectrum for three 2D EMHD turbulence runs, where we choose $n = 8$ to amplify the effect, $\eta_8 = 10^{-32}, 10^{-36}, 10^{-40}$ using $N^2 = 1024^2, 2048^2, 4096^2$, respectively. While there is no visible bottleneck effect for $\eta_8 = 10^{-32}$, it becomes more and more pronounced with decreasing dissipation coefficient leading to a flattening of the spectrum E_k (i.e., a steepening of the compensated spectrum $k^{5/3}E_k$ in Fig. 3) in an increasingly larger fraction of the inertial range, $k > \kappa \sim 60$.

This behavior is not limited to 2D EMHD turbulence, but is found to occur in a similar and even clearer form in 2D MHD turbulence simulations. Here the dynamical equations are

$$\partial_t\psi + \mathbf{v} \cdot \nabla\psi = -\eta_n(-\nabla^2)^n\psi\,, \quad (4)$$

$$\partial_t\omega + \mathbf{v} \cdot \nabla\omega - \mathbf{B} \cdot \nabla j = -\nu_n(-\nabla^2)^n\omega\,, \quad (5)$$

where $\mathbf{v} = \mathbf{e}_z \times \nabla\phi$ is the plasma flow (note that in spite of the formal similarity EMHD does *not* converge to MHD for $d_e \ll 1$. EMHD is limited to $d_e > \sqrt{m_e/m_i}$, while MHD is valid only at larger, macroscopic scales; see [15]). Previous numerical studies of decaying 2D MHD turbulence have revealed the spectral law $E_k \sim (v_A\epsilon)^{1/2}k^{-3/2}$ (see, e.g., [14]), where $v_A = B/\sqrt{4\pi\rho}$ is the Alfvén speed and the modal energy is $E_k = \sum_{\text{angle}} k^2(|\psi_k|^2 + |\phi_k|^2)$. The $k^{-3/2}$ spectrum results from the Alfvén effect [18], the coupling of small-scale velocity, and magnetic field fluctuations by the magnetic field of the large-scale eddies. For normal diffusion no bottleneck is discernable in the energy spectra [12,14]. To investigate this point more closely, we choose again a high-order dissipation operator in order to amplify the bottleneck effect, which may be hidden in the noise level for $n = 1$. Three simulation runs have been performed

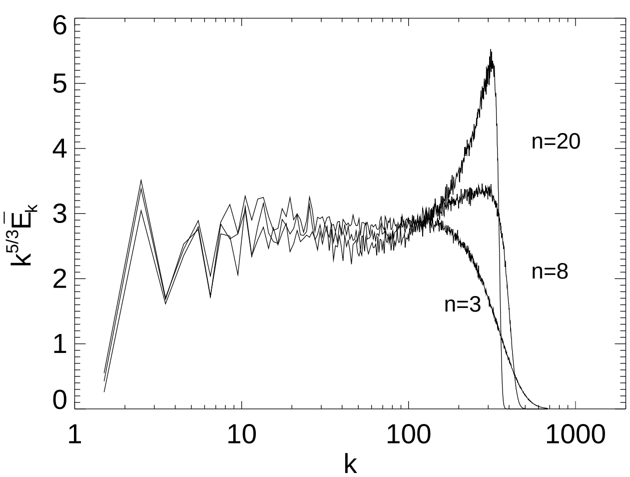

FIG. 1. Compensated energy spectra $k^{5/3}\overline{E}_k$ of 2D EMHD turbulence for diffusion operator order $n = 3, 8, 20$. Note the linear vertical scale.

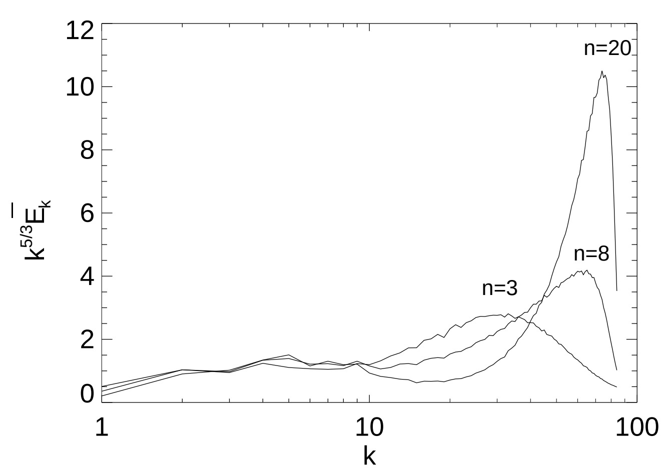

FIG. 2. Compensated energy spectra $\overline{E}_k$ for 3D EMHD turbulence simulations with $n = 3, 8, 20$. Normalization is such that the horizontal parts of the spectra coincide.

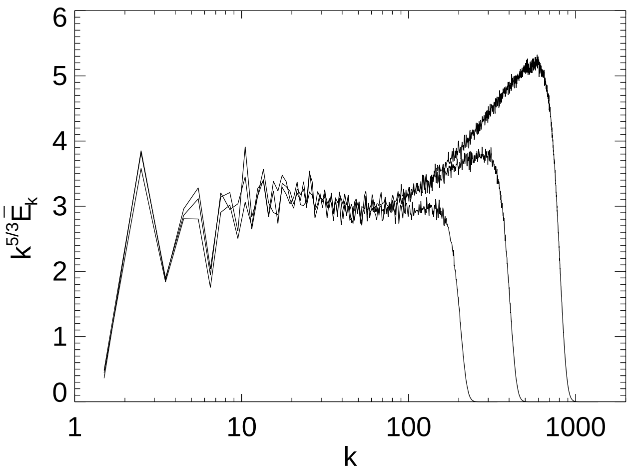

FIG. 3. Compensated energy spectra for three 2D EMHD turbulence simulations with $n = 8$, $\eta_8 = 10^{-32}, 10^{-36}, 10^{-40}$.

for decaying MHD turbulence using the same numerical scheme as in the EMHD simulations described above and similar initial conditions (called B-type in [12]). Figure 4 gives the time-averaged MHD energy spectra $\overline{E}_k = \langle E_k(v_A \epsilon)^{-1/2}\rangle_t$ plotted in compensated form for $\eta_8 = \nu_8 = 10^{-36}, 10^{-40}, 10^{-45}$ with resolutions $N^2 = 1024^2, 2048^2, 4096^2$, respectively. Comparing with the corresponding 2D EMHD cases given in Fig. 3, Fig. 4 shows a similar qualitative trend, a nonlocal influence of the dissipation range on the inertial range. The difference is probably due to the choice $d_e = 0.3$ in the EMHD runs in Fig. 3. Since a pure scaling behavior exists only for $kd_e > 1$, the effective scaling range is shorter by a factor of 3–4 for the same resolution, such that amplitude of the bottleneck pileup in the highest resolution EMHD case 4096^2 in Fig. 3 corresponds to the lowest-resolution MHD case 1024^2 in Fig. 4.

There is a nearly linear increase of the compensated spectrum in the log-linear plot, which suggests that

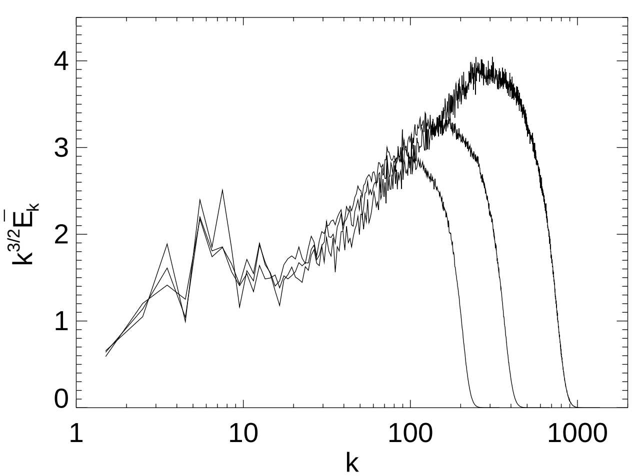

FIG. 4. Compensated 2D MHD energy spectra $k^{3/2}\overline{E}_k$ for $\eta_8 = \nu_8 = 10^{-36}, 10^{-40}, 10^{-45}$.

for sufficiently large $k > \kappa \sim 20$ the inertial range is modified by a logarithmic factor

$$E_k \sim k^{-3/2} \ln(k/\kappa)\,, \qquad (6)$$

where the magnitude of the effect is expected to depend on n, such that for $n = 1$ it becomes invisibly small at the achievable spatial resolution. The possibility of a logarithmic factor has been discussed in [6] for 3D turbulence, though only as a subdominant effect in the spectral correction term. The wave number κ is connected with some structure of the macrostate of the system. The fact that κ is lower in the MHD runs than observed in the EMHD runs is due to the choice $d_e = 0.3$ in the latter.

These results show that the bottleneck effect in the 2D turbulence systems considered, though very weak for normal diffusion $n = 1$, exhibits a nonlocal behavior when enhanced by choosing high n. The mechanism for this property must be connected with a stronger direct interaction of small- and large-scale modes in 2D than in 3D, which is also reflected in configuration space by the large-scale intermittency typical for 2D turbulence. The observed behavior is probably not due to the Alfvén effect, since the latter is not present in EMHD [15].

We would like to discuss briefly also a third type of turbulence, 2D thermal convection in the Boussinesq approximation described by the equations

$$\partial_t T + \mathbf{v} \cdot \nabla T + \partial_y \phi = -\chi_n(-\nabla^2)^n T\,, \qquad (7)$$

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega + \partial_y T = -\nu_n(-\nabla^2)^n \omega\,, \qquad (8)$$

again written in nondimensional form; see [19], where T is the temperature fluctuation and $\mathbf{v} = \mathbf{e}_z \times \nabla\phi$ the fluid velocity. Contrary to MHD or EMHD, where stationary turbulence can be achieved only by an external stirring force, this system is linearly unstable over a broad k range with growth rate $\gamma \propto k_y/k$, which generates a stationary level of turbulence. This is caused by the frozen-in mean temperature gradient T_0', $v_x T_0' = \partial_y \phi$ in the nondimensional form (7). Hence there is no ideal energy invariant; instead, one has

$$\frac{d}{dt}\int \frac{1}{2}(T^2 + v^2)\, d^2x = 2\int v_x T\, d^2x - \epsilon\,, \qquad (9)$$

where ϵ is the energy dissipation rate. Equations (7) and (8) have recently been studied numerically on a periodic box using a similar scheme as in the EMHD simulations. In spite of the anisotropic linear drive the spectra $E_k^T = |T_k|^2$ and $E_k^V = |v_k|^2$ are highly isotropic, which demonstrates the strong influence of the nonlinear terms. The energy dissipation rate $\epsilon = \epsilon_L^V + \epsilon_L^T + \epsilon_s^V + \epsilon_s^T$ in (9) consists of the dissipation on the velocity and the temperature fluctuations both at large (L) and small (s) wave numbers. The kinetic energy v^2 has an inverse cascade and is primarily dissipated at small wave numbers, where the modes are artificially damped to prevent condensation and suppression of turbulence [19], $\epsilon_L^V < \epsilon_s^V$, while the thermal fluctuation energy T^2 has a direct cascade, $\epsilon_L^T > \epsilon_s^T$.

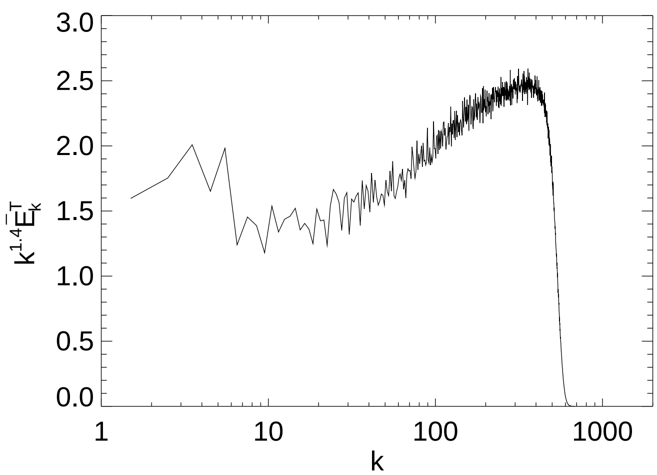

FIG. 5. 2D thermal convection. Compensated temperature fluctuation spectrum $k^{1.4}E_k^T$ for $n = 8$.

The results of [19] seem to indicate spectral laws $E_k^T \sim k^{-1.2}$ and $E_k^V \sim k^{-2.3}$, which differ somewhat from the expected Bolgiano scaling [20] $k^{-1.4}$ and $k^{-2.2}$, respectively. No convincing physical argument for these deviations could be given in [19] except the fact that the temperature fluctuations are found to be highly intermittent, which limits the relevance of the (nonintermittent) Bolgiano scalings $\delta_l T \sim l^{0.2}, \delta_l v \sim l^{0.6}$. Here we suggest an alternative interpretation of the simulation results in [19], namely that the deviations are caused by nonlocal effects, which should make the spectrum of the kinetic energy with an inverse cascade slightly steeper (cf. the 2D Euler case), while the spectrum of the temperature fluctuations, which have a direct cascade, should be flatter than the Bolgiano spectrum. For direct comparison with the EMHD and MHD simulations presented above, we have performed a similar run for thermal convection with $n = 8, \chi_8 = \nu_8 = 10^{-42}, N^2 = 2048^2$, from which Fig. 5 shows the temperature fluctuation spectrum compensated with the Bolgiano law. The behavior is indeed very similar to the MHD spectrum in Fig. 4.

In conclusion, we have investigated the spectral bottleneck pileup in 2D turbulence. While it is known that there is no such effect in the enstrophy cascade for 2D Euler turbulence, we have shown the existence of the effect in 2D systems exhibiting a direct energy cascade, in particular in MHD, EMHD, and thermal convection. The amplitude of the pileup is found to be significantly smaller than in corresponding 3D cases even for high-order hyperdiffusion, which enforces the tendency of spectral pileup by making the transition between inertial range and dissipative range more abrupt. But contrary to the local behavior in 3D the bottleneck effect in 2D turbulence has a nonlocal character, such that the major part of the inertial range is affected for sufficiently small dissipation coefficients. The simulation results suggest a logarithmic modification of the spectrum. This behavior reflects a strong direct interaction of disparate scales, which also manifests itself in the large-scale intermittency typically seen in configuration space plots of 2D turbulence. Also the deviations in the spectral laws from Bolgiano scaling in 2D thermal convection reported previously could be attributed to this nonlocal bottleneck effect. As a consequence the popular use of high-order diffusion operators in 2D turbulence simulations becomes rather doubtful.

Though the results presented in this Letter refer only to high-order diffusion, we believe that this behavior is generic for 2D turbulent systems with a direct energy cascade. In fact, recent very high-resolution studies of 2D MHD turbulence [21] using Newtonian dissipation $n = 1$ reveal a similar tendency, though bottleneck amplitudes are of course much smaller.

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