Approximating the Mapping between Systems Exhibiting Generalized Synchronization

Reggie Brown
Department of Physics and Department of Applied Science, College of William and Mary, Williamsburg, Virginia 23187-8795
(Received 5 May 1998; revised manuscript received 2 September 1998)

We present two methods for approximating the mapping between two systems exhibiting generalized synchronization. If the equations of motion are known then an analytic approximation to the mapping can be found. If time series data are used then a numerical approximation can be found. [S0031-9007(98)07833-8]

PACS numbers: 05.45.+b

The subject of synchronization between identical systems (denoted here by IS) has been of interest since the time of Huygens. Over the last decade it has become clear that even chaotic systems can be synchronized [1]. One example is drive-response synchronization, where

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \qquad \frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}) + \mathbf{E}(\mathbf{x},\mathbf{y}).$$

Here, $\mathbf{E}(\mathbf{x},\mathbf{y})$ denotes coupling between the drive system ($\mathbf{x}$) and the response system ($\mathbf{y}$). If $\mathbf{F}$ is deterministic, and if $\mathbf{E}(\mathbf{x},\mathbf{x}) = \mathbf{0}$, then the systems are synchronized where $\mathbf{y}(t_*) = \mathbf{x}(t_*)$. Because of determinism this condition remains true for $t > t_*$.

Recently, papers discussing a more general idea of synchronization have appeared in the literature. Drive-response dynamics for this type of synchronization is given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \qquad \frac{d\mathbf{y}}{dt} = \mathbf{G}(\mathbf{y};\mathbf{x}), \tag{1}$$

where $\mathbf{G}$ and $\mathbf{F}$ are permitted to be different functions. In principle, $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{y} \in \mathbb{R}^r$, and the dynamics takes place in $\mathbb{R}^{d+r}$. Intuitively, generalized synchronization (GS) occurs if the response state, $\mathbf{y}$, is related to the drive state, $\mathbf{x}$, by a time independent function, $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$. If GS occurs then the dynamics takes place on a d dimensional invariant manifold in $\mathbb{R}^{d+r}$.

Much of the work on GS has focused on three areas. The first area focuses on defining GS. Various definitions have been proposed [2–4]. Reference [4] suggests that subtleties associated with unstable periodic orbits imply that more than one definition may be required. The second area focuses on mathematical properties of $\boldsymbol{\phi}$. Rigorous results about the smoothness of $\boldsymbol{\phi}$, and the relationship between smoothness and Lyapunov exponents exist [3,5]. Also, numerical methods for determining the properties of $\boldsymbol{\phi}$ exist [6]. Since GS has been observed in experimental systems [7] it is structurally stable. Mathematical literature regarding the existence, stability, and smoothness of invariant manifolds is also relevant [8]. The last major area of research has focused on detecting GS from time series data [9,10]. The methods are indirect in the sense that they either do not approximate $\boldsymbol{\phi}$ or the approximations are local (often resulting in as many approximations as date points).

This manuscript opens a new research direction. Instead of seeking properties of $\boldsymbol{\phi}$, or indirect evidence of its existence, we believe it is better to go after the function itself. Therefore, we present methods for analytically and/or numerically constructing a single smooth function which globally approximates $\boldsymbol{\phi}$. If the equations of motion are known then an analytic approximation for $\boldsymbol{\phi}$ can be obtained. (To our knowledge, this is the only technique for analytically approximating $\boldsymbol{\phi}$.) The numerical method uses time series from the two systems to calculate a statistic which can be used to infer the existence of stable GS. The numerical method also gives a global approximation for the function, $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$. (We argue, implicitly, that if $\boldsymbol{\phi}$ and/or $\boldsymbol{\phi}^{-1}$ exist but are not well approximated by smooth functions then their usefulness is limited since their mathematical properties are probably "so bad" they prohibit most applications of GS.)

An important application for GS comes from control theory. Typically, control schemes work better when the complete state of the plant is known. The application uses measurements from the plant ($\mathbf{F}$) as drive input to an approximate model of the plant ($\mathbf{G}$). If GS occurs then the state of the plant can be approximated from the state of the model via $\mathbf{x} = \boldsymbol{\phi}^{-1}(\mathbf{y})$. This, and most other applications, requires a stable GS manifold.

Recently, several criteria have appeared for designing coupling which results in a stable IS manifold [11,12]. We report here that it is straightforward to show that a criteria for linearly stable GS is [13]

$$\mathbf{A} \equiv \langle \mathbf{D_y G}[\boldsymbol{\phi}(\mathbf{x});\mathbf{x}]\rangle,$$
$$-\mathrm{Re}[\Lambda_1] > \langle \| \mathbf{P}^{-1}[\mathbf{D_y G}[\boldsymbol{\phi}(\mathbf{x});\mathbf{x}] - \langle \mathbf{D_y G}\rangle]\mathbf{P}\| \rangle.$$

Here, $\langle \bullet \rangle$ denotes a time average over the driving trajectory, $\mathrm{Re}[\Lambda_1]$ is the eigenvalue of $\mathbf{A}$ with the largest real part, and $\mathbf{P}$ is a matrix whose columns are the eigenvectors of $\mathbf{A}$. Also, $\mathbf{D_y G}$ denotes the Jacobian of $\mathbf{G}$ with respect to $\mathbf{y}$. This criteria implies that if $\boldsymbol{\phi}$ and $\boldsymbol{\phi}^{-1}$ are known then one can estimate the state of the plant ($\mathbf{x}$) from the state of the model ($\mathbf{y}$) by design coupling which guarantees stable GS.

The analytical method used to approximate $\boldsymbol{\phi}$ is based on approximating center manifolds. Although the application to GS is new, complete discussions about approximating center manifolds (with examples) can be found in many textbooks [14]. Therefore, our discussion will be brief.

Assume the drive and response systems are given by Eq. (1). Taking the total time derivative of $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$, and using Eq. (1), implies that

$$\mathbf{G}[\boldsymbol{\phi}(\mathbf{x}); \mathbf{x}] - [\mathbf{D_x}\boldsymbol{\phi}(\mathbf{x})] \cdot \mathbf{F}(\mathbf{x}) = 0 \tag{2}$$

on the synchronization manifold. Here $\mathbf{D_x}\boldsymbol{\phi}$ is the Jacobian of $\boldsymbol{\phi}$. Equation (2) is interpreted as a partial differential equation for the unknown function, $\boldsymbol{\phi}(\mathbf{x})$. The same type of equation arises when estimating center manifolds [14].

Typically, Eq. (2) cannot be solved exactly. Therefore, approximate the solution by the series $\boldsymbol{\phi}(\mathbf{x}) = \mathbf{A} + \mathbf{B} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{M} \cdot \mathbf{x} + \cdots$. Next, insert the series into Eq. (2) and rewrite the results as a polynomial in powers of $\mathbf{x}$. The coefficients of this polynomial are functions of the parameters of $\mathbf{F}$ and $\mathbf{G}$ as well as the elements of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, etc. Also, this polynomial must hold for all $\mathbf{x}$ on the driving trajectory. If this trajectory is not a fixed point then it is reasonable to assume that the polynomial can hold only if the coefficient of each power of $\mathbf{x}$ vanishes. By equating each coefficient to zero we form a set of algebraic equations involving the parameters of $\mathbf{F}$, $\mathbf{G}$, and the elements of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, etc. The approximation to $\boldsymbol{\phi}(\mathbf{x})$ is obtained by solving these algebraic equations for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, etc., in terms of the parameters of $\mathbf{F}$ and $\mathbf{G}$.

Although conceptually straightforward, performing this procedure on anything but the simplest examples is very tedious, and soon grows beyond what can be done by hand. However, these calculations are not beyond the power of modern symbolic manipulation software. Indeed, the results presented below were obtained using MAPLE [15]. It was relatively straightforward to write a MAPLE program which produced these answers. Once the program was written the total run time was less than 10 min.

The approximation that one obtains for the GS manifold should hold near the attractor for the drive dynamics; however, it is not likely to be globally well defined. Although the results will not be presented, we have used a similar analysis to approximate $\mathbf{x} = \boldsymbol{\phi}^{-1}(\mathbf{y})$ for all of the examples discussed below.

If the GS manifold is stable then we can numerically approximate $\boldsymbol{\phi}$ from time series data. The numerical method used to approximate $\boldsymbol{\phi}$ is similar to one used by several authors to make empirical global models from time series data. Begin by assuming one has two data sets, $\mathbf{x}(n\Delta t) \in \mathbb{R}^d$ and $\mathbf{y}(n\Delta t) \in \mathbb{R}^r$, with $n = 1, 2, \ldots, N$, which represent simultaneous measurements of the drive and response systems at a sampling rate Δt. (If necessary, vector representations of the dynamics can be obtained from scaler time series via embedding techniques [16].) A measure for the dynamics of the drive system can be approximated by [16]

$$\rho(\mathbf{z}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta[\mathbf{z} - \mathbf{x}(n)].$$

Since the exact functional form of $\boldsymbol{\phi}$ is unknown, the best one can hope for is a series expansion

$$\boldsymbol{\phi}(\mathbf{z}) = \lim_{\mathbf{K}\to\infty} \sum_{\mathbf{I}=0}^{\mathbf{K}} \mathbf{p}^{(\mathbf{I})} \pi^{(\mathbf{I})}(\mathbf{z}). \tag{3}$$

Here, the $\mathbf{p}^{(\mathbf{I})}$'s are r dimensional expansion coefficients, which must be determined, and the $\pi^{(\mathbf{I})}(\mathbf{z})$'s represent some set of basis functions. Several authors have demonstrated the advantage of using a basis set which is orthonormal on $\rho(\mathbf{z})$, and they show how to construct such a basis from data using Gramm-Schmidt [17,18]. The summation index, $\mathbf{I}$ is used to identify the individual basis functions.

Once the basis set has been constructed, each expansion coefficient, $\mathbf{p}^{(\mathbf{I})}$, can be obtained by multiplying both sides of Eq. (3) by $\pi^{(\mathbf{I})}(\mathbf{z})\,\rho(\mathbf{z})$ and integrating over all space. Because of the orthonormality of the basis set we obtain

$$\mathbf{p}^{(\mathbf{I})} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}(n) \pi^{(\mathbf{I})}[\mathbf{x}(n)], \tag{4}$$

where we have used $\mathbf{y}(n) = \boldsymbol{\phi}[\mathbf{x}(n)]$ on the GS manifold. Thus, Eqs. (3) and (4) are used to approximate $\boldsymbol{\phi}$ from time series $\mathbf{x}(n)$ and $\mathbf{y}(n)$.

The last task is to determine the order at which to truncate the series in Eq. (3) so as to not overfit the data. This is done by using the minimum description length (MDL) criteria. These criteria are similar to the maximum likelihood principle associated with least squares fitting of data [18,19]. However, unlike maximum likelihood, MDL is capable of determining the optimal order at with to truncate Eq. (3). The MDL function we use is given by

$$\begin{aligned}\chi^2_{\mathrm{MDL}} = {} & \frac{rN}{2}[\ln(2\pi\hat{\sigma}^2) + 1] \\ & + N_p\left[\frac{1}{2} + \ln(\gamma)\right] \\ & - \ln(\eta) - \sum_{\mathbf{I}=\mathbf{0}}^{\mathbf{K}} \sum_{\beta=1}^{r} \ln(\delta_\beta^{(\mathbf{I})}).\end{aligned}$$

(See Ref. [19] for a complete derivation of this function.) Except for a positive constant (which we neglect), the first term is the usual prediction error from the maximum likelihood principle. Indeed, $\hat{\sigma}^2$ is the least squares prediction error obtained when predicting the $\mathbf{y}(n)$'s from the $\mathbf{x}(n)$'s.

The remaining terms are penalties which increase as more terms in Eq. (3) are retained and the model becomes more complex. N_p is the total number of nonzero $\mathbf{p}^{(\mathbf{I})}$'s retained in Eq. (3). In our implementation, a component of $\mathbf{p}^{(\mathbf{I})}$'s is set to zero if its statistical significance is not distinguishable from zero [20]. $\delta_\beta^{(\mathbf{I})}$ is the relative accuracy of the β component of $\mathbf{p}^{(\mathbf{I})}$, η is the relative accuracy of $\hat{\sigma}^2$, and $\gamma = 32$.

To illustrate the analytical and numerical techniques we applied them to examples using the Lorenz equations

$$\frac{dx_1}{dt} = s(x_2 - x_1),$$
$$\frac{dx_2}{dt} = rx_1 - x_1x_3 - x_2, \tag{5}$$
$$\frac{dx_3}{dt} = x_1x_2 - bx_3,$$

as the drive system.

The coupling between drive and response systems usually involves one of two cases. The first case arises when the physical processes responsible for the coupling are known so one has an explicit equation for the coupling. For this case one solves Eq. (2) as discussed above. Below we consider the second case where the response system is given by $\mathbf{G}(\mathbf{y}) + \mathbf{E}[\boldsymbol{\phi}(\mathbf{x}) - \mathbf{y}]$. Here, the coupling obeys $\mathbf{E}(\mathbf{0}) = \mathbf{0}$ and is used to ensure that the GS manifold is stable. The problem with this case is that we cannot evaluate $\mathbf{E}[\boldsymbol{\phi}(\mathbf{x}) - \mathbf{y}]$ because we do not know, *a priori*, the form of $\boldsymbol{\phi}(\mathbf{x})$. For the examples discussed below this problem is overcome by calculating $\boldsymbol{\phi}$ in two stages.

In the first stage we calculate $\boldsymbol{\phi}$ using diagonal coupling $\mathbf{E}[\boldsymbol{\psi}(\mathbf{x}) - \mathbf{y}] \equiv \epsilon[\boldsymbol{\psi}(\mathbf{x}) - \mathbf{y}]$ where $\boldsymbol{\psi}$ is an arbitrary function. The $\boldsymbol{\phi}$ calculated in this first stage clearly depends on $\boldsymbol{\psi}$. In the second stage we force $\boldsymbol{\psi}(\mathbf{x}) = \boldsymbol{\phi}(\mathbf{x})$. This second stage ensures that $\mathbf{E}[\boldsymbol{\psi}(\mathbf{x}) - \mathbf{y}] = 0$ on the GS manifold.

Two trivial tests of the analytic method involved defining $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x}) = [x_1 + \alpha x_2 + \beta x_2^2, x_2, x_3]$ for one test, and $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x}) = [x_1 + \alpha x_3^2, \beta x_2, x_3 + \gamma x_2^2]$ for the other. For each test we obtain a response system, $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$, by taking the time derivative of $\mathbf{y}$, using Eq. (5) to resolve the vector field, $\mathbf{G}(\mathbf{y})$, and adding diagonal coupling. For this test, the response systems are the Lorenz system after a nonlinear change of coordinates, and the analytic method easily recovered the GS manifolds, $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$.

A final test of the analytic procedure used the following response system:

$$\frac{dy_1}{dt} = s(1 + \delta)(y_2 - y_1),$$
$$\frac{dy_2}{dt} = r(1 + \Delta)y_1 - y_1y_3 - y_2, \tag{6}$$
$$\frac{dy_3}{dt} = y_1y_2 - b(1 + \eta)y_3,$$

with diagonal as discussed above.

For this example we could only approximate $\boldsymbol{\phi}$. The approximation contained three arbitrary constants, thus it is not unique. We selected values for two of them so that $\boldsymbol{\phi}$ has a simple form. [For the trivial examples discussed above this choice always led to the "correct" equation for $\boldsymbol{\phi}(\mathbf{x})$]. The third constant appears trivially in B_{33}, is of order (Δ, δ, η), and is denoted by K below. Finally, the approximation is simplified by retaining terms that are second order in $\mathbf{x}$, first order in (δ, Δ, η), and in the limit of large coupling strength, ϵ. (Thus, we examine a case where the response system is close to the drive system.)

With these criteria in mind we found that $\boldsymbol{\phi}(\mathbf{x})$ is given by $\mathbf{A} = \mathbf{0}$,

$$\mathbf{B} = \mathbf{1} + \frac{1}{\Gamma}\begin{bmatrix} -(r^2\Delta - s^2\delta) & -s(r + 1 - s)\delta & 0 \\ -[r(s - 1)\Delta - s^2\delta] & (r^2\Delta - s^2\delta) & 0 \\ 0 & 0 & K \end{bmatrix},$$

where $\mathbf{1}$ is the identity matrix, and the three tensor $\mathbf{M}$ is given by $\mathbf{M}^{(1)} = \mathbf{M}^{(2)} = \mathbf{0}$, and

$$\mathbf{M}^{(3)} = \frac{1}{\Gamma}\begin{bmatrix} -\frac{r(s-1)\Delta - s^2\delta}{b-2s} & 0 & 0 \\ 0 & \frac{s(r+1-s)\delta}{b-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In these equations, $\Gamma = (2r^2 + 3s^2 - 2s + 1)$. Furthermore, it clear that this transformation satisfies $\boldsymbol{\phi} = \mathbf{1}$ in the limit $\delta, \Delta, \eta \to 0$.

To test the numerical method we first demonstrate that it can determine the correct form of $\boldsymbol{\phi}$ for stable GS from time series data. To accomplish this we used Eq. (5) (with $s = 16$, $b = 4$, and $r = 46$) as the drive system

TABLE I. Numerical approximations for the transformation $\boldsymbol{\phi}_1 = x_1 - 0.01x_3^2$, $\boldsymbol{\phi}_2 = 0.95x_2$, and $\boldsymbol{\phi}_3 = x_3 + 0.03x_2^2$. If the calculated value of $\boldsymbol{\phi}_j$ was of order 10^{-5} or less then it was set to zero.

	$\sigma = 0$			$\sigma = 0.05$		
Factor	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}_3$	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}_3$
Const	0.001 01	0.000 55	−0.001 94	0.0822	0.002 15	0.004 75
x_1	1.00	0	0	1.023	0	−0.0390
x_2	0	0.950	0	−0.0107	0.954	0.0147
x_3	0	0	1.00	−0.0061	0	1.003
x_1x_1	0	0	0	0.000 982	0	0.000 166
x_1x_2	0	0	0	−0.000 304	0	−0.000 259
x_1x_3	0	0	0	−0.000 233	0	0.000 381
x_2^2	0	0	0.0300	0	0	0.0297
x_2x_3	0	0	0	0	0	0
x_3^2	−0.0100	0	0	−0.009 94	0	0

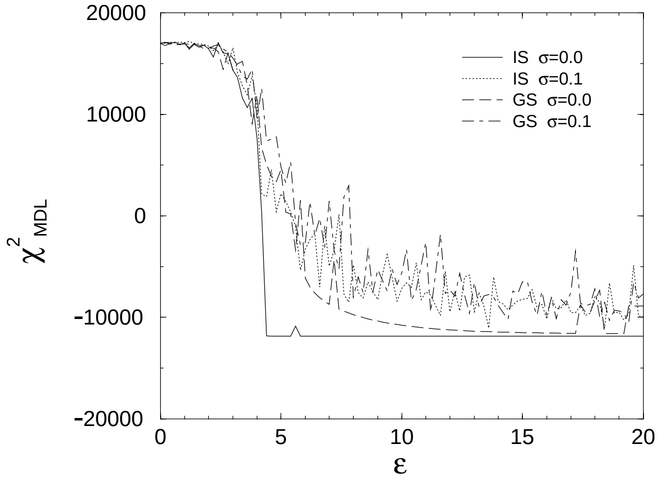

FIG. 1. The sudden drop at $\epsilon \simeq 4$ indicates the onset of stable synchronization and stable generalized synchronization.

and a response system obtained from $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x}) = [x_1 - 0.01x_3^2,\ 0.95x_2, x_3 + 0.03x_2^2]$. The systems were coupled via the y_2 equation using $\epsilon[(0.95x_2 + \text{noise}) - y_2]$. The noise was Gaussian white with zero mean and standard deviation, 15σ. Here, 15 is approximately the standard deviation of x_2, and $\sigma = 0$ or 0.05. $\epsilon = 10$ was used because, with y_2 coupling and a chaotic driving trajectory, IS is "stable" for $\epsilon \geq 4$ [11].

The numerical procedure was given $N = 4000$ simultaneously recorded values of **x** and **y** at a sampling interval of $\Delta t = 0.02$. The results (see Table I) indicate that the numerical procedure found a good approximation to $\boldsymbol{\phi}$, even in the presence of small amounts of noise.

To further test the numerical method we used Eqs. (5) and (6) (the same values for s, b, and r) and a drive signal, $\epsilon\{[\boldsymbol{\phi}_2(\mathbf{x}) + \text{noise}] - y_2\}$, coupled to the y_2 equation. These tests used simultaneously recorded scalar time series of the same length and sampling interval given above. Scalars were obtained using the arbitrarily chosen projections

$$s_d(n) = x_1(n) - 2.5x_2(n) + 0.75x_3(n)\,,$$
$$s_r(n) = -0.5y_1(n) + 1.5y_2(n) - y_3(n)\,.$$

Each scalar time series was independently rescaled to mean zero and standard deviation one, and an attractor for each time series was reconstructed using a time delay embedding [16].

The results of our attempts to approximate $\boldsymbol{\phi}$ for $\delta = \Delta = \eta = 0$ (IS) and $\delta = 0.02$, $\Delta = 0.04$, $\eta = -0.03$ (GS) are shown in Fig. 1. The figure shows that χ^2_{MDL} experiences a sharp drop at $\epsilon \simeq 4$ when the drive/response systems are identical and a less sharp drop for GS. The drop implies that the numerical procedure has found a relatively accurate approximation for $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$, so the GS manifold is stable. Also, the figures show that the procedure deteriorates gracefully in the presence of noise.

In conclusion, we have presented an analytical and a numerical method for approximating the mapping that defines the invariant manifold associated with generalized synchronization.

The author thanks Dr. N. F. Rulkov, Dr. L. M. Pecora, Dr. B. R. Hunt, and Dr. J. Stark for valuable discussions and comments that led to this work. This work was supported by the National Science Foundation under CAREER Grant No. PHY-972236, and the Air Force Office of Scientific Research under Grant No. F49620-98-1-0144.

[1] See Chaos **7** (1997), and references therein.
[2] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Radiophys. Quantum Electron. **29**, 747 (1986).
[3] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996); B. R. Hunt, E. Ott, and J. A. Yorke, Phys. Rev. E **55**, 4029 (1997).
[4] U. Parlitz, L. Junge, and L. Kocarev, Phys. Rev. Lett. **79**, 3158 (1997).
[5] J. Stark, Physica (Amsterdam) **109D**, 163 (1997).
[6] L. M. Pecora, T. L. Carroll, and J. F. Heagy, Phys. Rev. E **52**, 3420 (1995); K. Pyragas, Phys. Rev. E **54**, R5408 (1996).
[7] N. F. Rulkov and M. M. Sushchik, Phys. Lett. A **214**, 145 (1996); A. Kittel, J. Parisi, and K. Pyragas, Physica (Amsterdam) **112D**, 459 (1998).
[8] K. Josic, Phys. Rev. Lett. **80**, 3054 (1998); R. J. Sacker, J. Math. Mech. **18**, 705 (1969); N. Fenichel, Ind. Univ. Math. J. **21**, 193 (1971).
[9] S. J. Schiff, P. So, T. Chang, R. E. Burke, and T. Sauer, Phys. Rev. E **54**, 6708 (1996).
[10] H. D. I. Abarbanel, N. F. Rulkov, and M. M. Sushchik, Phys. Rev. E **53**, 4528 (1996); N. F. Rulkov, M. M. Sushchik, L. S. Tsimring and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).
[11] R. Brown and N. F. Rulkov, Chaos **7**, 395 (1997).
[12] D. J. Gauthier and J. C. Bienfang, Phys. Rev. Lett. **77**, 1751 (1996); D. M. Walker and A. I. Mees, Centre for Applied Dynamics and Optimisation report, University of Western Australia.
[13] This result is easily obtained by applying methods discussed in Ref. [11].
[14] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* (Springer-Verlag, New York, NY, 1983).
[15] A. Heck, *Introduction to Maple,* (Springer-Verlag, New York, NY, 1996), 2nd ed.
[16] H. D. I. Abarbanel, R. Brown, J. J. Sidorowich, and L. S. Tsimring, Rev. Mod. Phys. **65**, 1331 (1993).
[17] M. Geona, F. Lentini, and V. Cimagalli, Phys. Rev. A **44**, 3496 (1991); C. Letellier, L. LeSceller, E. Marechal, P. Dutertre, B. Maheu, G. Gouesbet, Z. Fei, and J. L. Hudson, Phys. Rev. E **51**, 4262 (1995).
[18] R. Brown, N. F. Rulkov, and E. R. Tracy, Phys. Rev. E **49**, 3784 (1994).
[19] K. Judd and A. Mees, Physica (Amsterdam) **82D**, 426 (1995).
[20] R. Brown, V. In, and E. R. Tracy, Physica (Amsterdam) **102D**, 208 (1997).