Zeta Functions, Renormalization Group Equations, and the Effective Action

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We demonstrate how to extract all the one-loop renormalization group equations for *arbitrary* quantum field theories from knowledge of an appropriate Seeley-DeWitt coefficient. By formally solving the renormalization group equations to one loop, we renormalization group *improve* the classical action and use this to derive the leading logarithms in the one-loop effective action for arbitrary quantum field theories. [S0031-9007(98)07781-3]

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It is well known that *any* quantum field theory (QFT) can be renormalized to one loop via the magic of zeta function techniques [1–7]. These techniques are so powerful that they hide all of the divergence structure in the woodwork, which has caused the community to miss a very important point: The one-loop renormalization group equations (RGE's) for *arbitrary* QFT's can be extracted from knowledge of an appropriate Seeley-DeWitt coefficient. Assuming only that the kinetic energy terms are quadratic in derivatives, and that spacetime has d dimensions, the appropriate Seeley-DeWitt coefficient is $a_{d/2}$, which governs the conformal anomaly [1,6,7]. For our purposes the central observation is that $a_{d/2}$ also governs the one-loop logarithmic divergences and so controls the running of the coupling constants at one-loop order. By expanding the classical action and Seeley-DeWitt coefficients in terms of primitive symmetry invariants, the one-loop RGE's can easily (if formally) be written down for arbitrary QFT's. It is then easy to see that *all* one-loop beta functions vanish in odd-dimensional spacetimes, so that there is no running of the coupling constants at one-loop order. For even-dimensional spacetimes, with an appropriate set of conventions, all beta functions can be written as homogeneous multinomials of order $d/2$.

By formally solving the RG equations, we can one-loop *improve* the classical action, which allows us to extract information about the one-loop effective action without ever resorting to explicit Feynman diagram calculations. In particular, the form of the leading-logarithmic contributions to the effective action is completely specified in terms of the $a_{d/2}$ Seeley-DeWitt coefficient. These coefficients are tabulated in many places, and computation is now essentially automated [8].

Effective action and RGE at one loop.—Consider an arbitrary quantum field $\phi(\vec{x}, t)$ governed by a classical action $S[\phi, \lambda_i]$. The field ϕ may be scalar, spinor, or (with suitable caveats) even a gauge field. The set $\{\lambda_i\}$ denotes the complete set of (generalized) coupling constants in the theory. It is a standard result that (in terms of bare quantities) the one-loop effective action is

$$\Gamma[\phi;\phi_0] = S[\phi,\lambda_i] - S[\phi_0,\lambda_i] + \frac{\hbar}{2}\,\mathrm{Str}\left\{\ln\det\frac{S_2[\phi,\lambda_i]}{\mu^2} - \ln\det\frac{S_2[\phi_0,\lambda_i]}{\mu^2}\right\} + O(\hbar^2)\,. \tag{1}$$

Here Str denotes a "supertrace," a sum over all Bose and Fermi fields in the theory with a + sign for Bose fields and a − for Fermi fields. Spin degeneracy factors are subsumed into the determinant. $S_2[\phi]$ denotes $\delta^2 S[\phi]/\delta\phi(x)\delta\phi(y)$, which is a second-order differential operator that governs the Gaussian fluctuations. (For a unified notation, fermion determinants can always be converted to second order by squaring before taking the determinant. Also, for gauge fields one should be careful to include terms due to gauge breaking, and the unitarity preserving ghosts [9,10].) The arbitrary parameter μ has been introduced for purely dimensional reasons (to keep the argument of the logarithm dimensionless). Furthermore S_2 is second order because in this Letter we are interested in QFT's; for the considerably more complicated nonquantum field theories associated with stochastic differential equations [11] this particular assumption must be modified [12]. Finally ϕ_0 is some suitable background field (a classical solution of the equations of motion for zero source), typically a minimum of the bare potential, a zero gauge field strength, Minkowski spacetime, or even Schwarzschild spacetime. The above is, of course, a divergent quantity which has to be regularized and renormalized. Invoking completely standard machinery, to be found in many QFT textbooks [11,13,14], we do so with the result that (now in terms of renormalized quantities)

$$\Gamma[\phi;\phi_0] = S[\phi,\lambda_i(\mu)] - S[\phi_0,\lambda_i(\mu)] + \frac{\hbar}{2}\,\mathrm{Str}\left\{\ln\det\frac{S_2[\phi,\lambda_i(\mu)]}{\mu^2} - \ln\det\frac{S_2[\phi_0,\lambda_i(\mu)]}{\mu^2}\right\} + O(\hbar^2)\,. \tag{2}$$

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The coupling constants now in general "run" with the renormalization scale μ. (The only slightly nonstandard thing we have done here is to avoid using the wave function renormalization $Z(\mu)$ to rescale the quantum field; instead we find it more convenient to view wave function renormalization as just another coupling constant.) The *exact* renormalization group equations are simply the statement that the effective action does not depend on the renormalization scale, i.e.,

$$\frac{\mu d\Gamma[\phi;\phi_0]}{d\mu} = 0\,. \tag{3}$$

Now from zeta function technology (or with a little more work from any other regularization and renormalization scheme) and assuming for simplicity the lack of infrared divergences, we have the *exact* mathematical result that [1–7]

$$\ln\det\frac{S_2[\phi,\lambda_i(\mu)]}{\mu^2} = \ln\det\frac{S_2[\phi,\lambda_i(\mu)]}{\mu_0^2} + \frac{1}{(4\pi)^{d/2}} \times \int d^dx\, a_{d/2}[\phi,\lambda_i(\mu)]\ln\left(\frac{\mu}{\mu_0}\right). \tag{4}$$

To be even more explicit, this entails

$$\ln\det\frac{S_2[\phi,\lambda_i(\mu)]}{\mu^2} = \ln\det\frac{S_2[\phi,\lambda_i(\mu_0)]}{\mu_0^2} + \frac{1}{(4\pi)^{d/2}} \times \int d^dx\, a_{d/2}[\phi,\lambda_i(\mu)]\ln\left(\frac{\mu}{\mu_0}\right) + O(\hbar)\,. \tag{5}$$

So, inserting this into the exact RGE we obtain to one-loop order,

$$\frac{\mu dS[\phi,\lambda_i(\mu)]}{d\mu} - \frac{\mu dS[\phi_0,\lambda_i(\mu)]}{d\mu} = -\frac{\hbar}{2(4\pi)^{d/2}}\int d^dx\,\mathrm{Str}\{a_{d/2}[\phi,\lambda_i(\mu)] - a_{d/2}[\phi_0,\lambda_i(\mu)]\} + O(\hbar^2)\,. \tag{6}$$

Equivalently

$$\left\{\frac{dS[\phi,\lambda_i(\mu)]}{d\lambda_i(\mu)} - \frac{dS[\phi_0,\lambda_i(\mu)]}{d\lambda_i(\mu)}\right\}\frac{\mu d\lambda_i(\mu)}{d\mu} = -\frac{\hbar}{2(4\pi)^{d/2}}\int d^dx\,\mathrm{Str}\{a_{d/2}[\phi,\lambda_i(\mu)] - a_{d/2}[\phi_0,\lambda_i(\mu)]\} + O(\hbar^2)\,. \tag{7}$$

Extracting beta functions from this is now completely straightforward: We pick off terms of the same functional form from both sides of the above [15–17]. Results may be simplified drastically by choosing an appropriate set of conventions. Let Φ_i be a basis of elementary terms in the classical action constrained only by symmetry. For instance, for a scalar theory we would typically have $\Phi_0 = \frac{1}{2}(\partial\phi)^2$ and $\Phi_n = \frac{1}{n!}\phi^n$, for fermions we would take $\Phi_0 = \overline{\psi}[\gamma^\mu(\partial_\mu - A_\mu)]\psi$, $\Phi_2 = m\overline{\psi}\psi$, and for gauge theories $\Phi_0 = F^2$ and $\Phi_1 = F\tilde{F}$. For mixed theories we just have to rearrange the indices, and without loss of generality we can adopt the convention that the action is linear in this basis and in the generalized coupling constants:

$$S[\phi,\lambda_i(\mu)] = \sum_i \lambda_i(\mu)\int d^dx\,\Phi_i\,. \tag{8}$$

This is only a convention; it is not a restriction on the class of theories considered. Given this choice of basis we can also expand the Seeley-DeWitt coefficient as

$$\mathrm{Str}(a_{d/2}[\phi,\lambda_i(\mu)]) = \sum_i \kappa_i(\lambda_j(\mu))\Phi_i\,. \tag{9}$$

That the same set of elementary terms can be used to expand both the classical action *and* the integrated Seeley-DeWitt coefficient is a consequence of renormalizability. Specifically, for renormalizable and superrenormalizable theories the counterterms are by definition equal to or fewer than the elementary terms in the classical action, which implies that the Seeley-DeWitt coefficient is expandable using the elementary terms occurring in the classical action as a basis. For nonrenormalizable theories this fails, since there are terms in the integrated Seeley-DeWitt coefficient that do not appear in the classical action. (The Seeley-DeWitt coefficient will often contain total derivatives, such as $\nabla^2\phi$, which could be added to the classical Lagrangian without affecting the classical action and so can be omitted altogether since we are only really interested in spacetime integrals.) With all these conventions in place, our (*slightly nonstandard*) one-loop beta functions are

$$\beta_i(\lambda_j) \stackrel{\mathrm{def}}{=} \frac{\mu d\lambda_i(\mu)}{d\mu} = -\frac{\hbar}{2(4\pi)^{d/2}}\,\kappa_i(\lambda_j(\mu)) + O(\hbar^2)\,. \tag{10}$$

An immediate consequence is that *all* of our beta functions vanish to one loop in odd-dimensional spacetimes, simply because the Seeley-DeWitt coefficient vanishes in odd-dimensional spacetimes [7]. (This is intimately related to the vanishing of the conformal anomaly in odd-dimensional spacetimes [18].) This statement is not limited to flat space and continues to hold true even for QFT's defined on curved spacetimes. It will however fail in general for manifolds with boundary where the classical action must contain both bulk and surface contributions. While the coupling constants associated with the bulk action do not run at one loop, those coupling constants appearing in the surface action will generally run at one loop. We are *not* asserting that all odd-dimensional (spacetime) theories are one-loop finite, but the much more modest claim that all odd-dimensional (spacetime) theories are one-loop nonrunning.

Explicit calculation quickly verifies that all one-loop beta functions vanish for model theories such as QED_3

and $\lambda(\phi^4)_3$. In contrast, in the ϵ-expansion, one-loop beta functions for $QED_{4-\epsilon}$ and $\lambda(\phi^4)_{4-\epsilon}$ do not vanish, but these beta functions should not be trusted for $\epsilon = 1$.

The attentive reader might profitably wonder where we have hidden all of the anomalous dimensions? All anomalous dimensions have been converted into beta functions via the schema

$$\gamma_Z \overset{\text{def}}{=} \frac{\mu d \ln Z(\mu)}{d\mu} \Rightarrow \beta_Z \overset{\text{def}}{=} \frac{\mu dZ(\mu)}{d\mu} = Z\gamma_Z . \quad (11)$$

Is there anything more we can say about the $\kappa_i(\lambda_j)$ without resorting to explicit calculations? Start by observing that the Jacobi field operator is by definition linear in the couplings,

$$S_2[\phi, \lambda_i(\mu)] = \sum_i \lambda_i(\mu) \frac{\delta^2 \Phi_i}{\delta\phi(x)\delta\phi(y)}\bigg|_\phi , \quad (12)$$

and note that the $(d/2)$th Seeley-DeWitt coefficient is homogeneous in the Jacobi field operator,

$$a_{d/2}(\alpha S_2) = a_{d/2}(S_2) . \quad (13)$$

This can be derived from the definition of $a_{d/2}$ in terms of the short-time expansion of the heat kernel and implies that for *all* QFT's at one loop,

$$\beta_i(\alpha\lambda_j) = \beta_i(\lambda_j) . \quad (14)$$

This property is often enough to completely pin down the form (if not the coefficients) of the one-loop beta functions. For instance, for pure gauge theories (no matter fields), our conventions imply that we must write $S(\phi, \lambda) = F^2/g^2$. Since there is only one coupling constant present in the theory ($\lambda_0 = 1/g^2$) homogeneity implies

$$\frac{\mu d(1/g^2)}{d\mu} = \hbar k + O(\hbar^2) . \quad (15)$$

Here k is now some constant independent of g. That is: Gauge symmetry plus the analysis of this Letter is enough to specify the *form* of the one-loop beta function completely.

Scalar field theory provides another useful example: In this case the coupling constant λ_0 attached to the kinetic energy term (Φ_0) plays a special role, and the homogeneity relation can always be used to scale it out of the $a_{d/2}$ coefficient. Furthermore, there are well-known recursion relations for calculating the Seeley-DeWitt coefficients in terms of the Jacobi field operator. The key point is that $a_{d/2}$ contains terms of the type $[(S_2)^{d/2} I(x, x')]$, plus lower powers of S_2, plus derivative terms that integrate to zero. [Here $I(x, x')$ is the parallel displacement operator, and the square brackets indicate the coincidence limit $x' \to x$.] This implies that $\int a_{d/2}$ is a multinomial in (λ_i/λ_0) of the order of $d/2$ (i.e., containing terms up to $\lambda_i^{d/2}$). More specifically, for two, four, and six dimensions, the beta functions (using our conventions) must always be of the form

$$\beta_i(\lambda_j; d = 2) = -\frac{\hbar}{8\pi} \kappa_i{}^j \frac{\lambda_j}{\lambda_0} + O(\hbar^2) , \quad (16)$$

$$\beta_i(\lambda_j; d = 4) = -\frac{\hbar}{32\pi^2} \kappa_i{}^{jk} \frac{\lambda_j}{\lambda_0}\frac{\lambda_k}{\lambda_0} + O(\hbar^2) , \quad (17)$$

$$\beta_i(\lambda_j; d = 6) = -\frac{\hbar}{128\pi^3} \kappa_i{}^{jkl} \frac{\lambda_j}{\lambda_0}\frac{\lambda_k}{\lambda_0}\frac{\lambda_l}{\lambda_0} + O(\hbar^2) , \quad (18)$$

with the obvious pattern holding for higher dimensions. This can be checked against explicit computations for standard theories (see, e.g., Collins [14] or [17]) which show that the κ's are simple rational numbers. For $\lambda(\phi^4)_4$ these constraints can be used to completely fix the *form* of the one-loop beta functions. This structure can also be justified via rather general Feynman diagram considerations: the beta functions at one loop are a reflection of the logarithmic divergences; in d dimensions we get one-loop logarithmic divergences only from a polygonal loop with $d/2$ propagators (and so $d/2$ vertices). With our conventions each one of these vertices must contain exactly one λ_i and each propagator a $1/\lambda_0$, which completes the proof.

RG improvement and consistency check.—Suppose now we have extracted the RGE's and have integrated them up to obtain $\lambda_i(\mu) = f_i(\lambda_j(\mu_0), \mu/\mu_0)$. The improved action is defined by inserting these running parameters into the classical action,

$$S_{\text{improved}}(\phi, \lambda_i(\mu)) \overset{\text{def}}{=} S(\phi, \lambda_i \to \lambda_i(\mu)) . \quad (19)$$

The RGE's [cf. Eq. (7)] have been carefully arranged so

$$S_{\text{improved}}(\phi, \lambda_i(\mu)) = S_{\text{improved}}(\phi, \lambda_i(\mu_0)) + \frac{\hbar}{2(4\pi)^{d/2}} \times \int d^d x \,\text{Str}\, a_{d/2}[\phi, \lambda_i(\mu)] \times \ln\left(\frac{\mu}{\mu_0}\right) + O(\hbar^2) . \quad (20)$$

Therefore to one-loop order

$$\Gamma[\phi;\phi_0] = S[\phi, \lambda_i(\mu)] - S[\phi_0, \lambda_i(\mu)] + \frac{\hbar}{2}\text{Str}\left\{\ln\det\frac{S_2[\phi, \lambda_i(\mu)]}{\mu^2} - \ln\det\frac{S_2[\phi_0, \lambda_i(\mu)]}{\mu^2}\right\} + O(\hbar^2) , \quad (21)$$

$$= S[\phi, \lambda_i(\mu_0)] - S[\phi_0, \lambda_i(\mu_0)] + \frac{\hbar}{2}\text{Str}\left\{\ln\det\frac{S_2[\phi, \lambda_i(\mu_0)]}{\mu_0^2} - \ln\det\frac{S_2[\phi_0, \lambda_i(\mu_0)]}{\mu_0^2}\right\} + O(\hbar^2) . \quad (22)$$

This verifies, as it should, that physics is independent of the choice of renormalization scale μ.

Leading logarithms.—The general solution of the RGE is

$$\Gamma[\phi;\phi_0] = S[\phi,\lambda_i(\mu)] - S[\phi_0,\lambda_i(\mu)] + \frac{\hbar}{2d(4\pi)^{d/2}} \times \int d^d x \left\{ \mathrm{Str}\left(a_{d/2}[\phi,\lambda_i(\mu)] \ln \frac{a_{d/2}[\phi,\lambda_i(\mu)]}{\mu^d} - a_{d/2}[\phi_0,\lambda_i(\mu)] \ln \frac{a_{d/2}[\phi_0,\lambda_i(\mu)]}{\mu^d} \right) + X[\lambda_i(\mu),\Phi_i] \right\} + O(\hbar^2) \,. \tag{23}$$

We use the fact that the RGE [Eq. (3) or equivalently Eq. (7)] is a quasilinear first-order partial differential equation [19] and adjust integration constants in a convenient way. (Some special cases are discussed in [17].) The integration constant X is constrained by the facts that (1) it *cannot* depend explicitly on μ, and (2) by dimensional analysis and renormalizability, to be of the form

$$X[\lambda_i(\mu),\Phi_i] = \sum_i \epsilon_i(\lambda_j(\mu),\Phi_j)\lambda_i(\mu)\Phi_i \,. \tag{24}$$

Here the $\epsilon_i(\lambda_j(\mu),\Phi_j)$ are dimensionless functions of the indicated variables. This is sometimes sufficient to specify the ϵ_i completely. For instance, for scalar field theories in the constant field limit (i.e., the effective potential) the ϵ_i are known to be constants [17] and so simply correspond to finite renormalization ambiguities. Thus Eq. (23) in this case provides the *exact* one-loop effective potential. This also holds for fermion plus scalar systems with Yukawa interactions, but once background gauge fields are switched on there are too many dimension full operators present to usefully constrain $X[\lambda_i,\Phi_i]$ [17].

More generally we can appeal to a variant of the decoupling theorem [14], by noting that $a_{d/2}$ behaves like a mass term for the Gaussian fluctuations. Thus an expansion in strong fields is equivalently an expansion in large masses and so the decoupling theorem justifies

$$\Gamma[\phi;\phi_0] = S[\phi,\lambda_i(\mu)] - S[\phi_0,\lambda_i(\mu)] + \frac{\hbar}{2d(4\pi)^{d/2}} \times \int d^d x \, \mathrm{Str}\left\{ a_{d/2}[\phi,\lambda_i(\mu)] \left[\ln \frac{a_{d/2}[\phi,\lambda_i(\mu)]}{\mu^d} + O\left(\frac{a_{d/2}[\phi_0,\lambda_i(\mu)]}{a_{d/2}[\phi,\lambda_i(\mu)]} \right) \right] \right\} + O(\hbar^2) \,. \tag{25}$$

In this Letter we have shown that essentially all of one-loop physics for all QFT's (i.e., systems with fluctuations governed by a second-order differential operator) can be extracted from the appropriate Seeley-DeWitt coefficient, $a_{d/2}$. The analysis also puts very strong constraints on the form of the one-loop beta functions without ever having to resort to a specific Feynman diagram calculation.

[1] J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976); **15**, 1484 (1977); J. S. Dowker, J. Phys. A **11**, 347 (1977).
[2] S. Hawking, Commun. Math. Phys. **56**, 133 (1977).
[3] G. Gibbons, in Ref. [6].
[4] B. S. DeWitt, in Ref. [6].
[5] S. W. Hawking, in Ref. [6].
[6] *General Relativity: an Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[7] S. K. Blau, M. Visser, and A. Wipf, Nucl. Phys. **B310**, 163 (1988).
[8] I. G. Avramidi and R. Schimming, in *Quantum Field Theory under the Influence of External Conditions,* edited by M. Bordag (Tuebner, Leipzig, 1996); hep-th/9510206; M. J. Booth, *HeatK: A Mathematica Program for Computing Heat Kernel Coefficients,* hep-th/9803113.
[9] G. K. Savvidy, Phys. Lett. **71B**, 133 (1977).
[10] I. G. Avramidi, J. Math. Phys. (N.Y.) **36**, 1557 (1995); Phys. Lett. B **305**, 27 (1993); see also I. G. Avramidi, in *Quantum Field Theory under the Influence of External Conditions* (Ref. [8]).
[11] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, England, 1996).
[12] D. Hochberg, C. Molina-París, J. Pérez-Mercader, and M. Visser (to be published).
[13] S. Weinberg, *The Quantum Theory of Fields I & II* (Cambridge University Press, Cambridge, England, 1996).
[14] J. Collins, *Renormalization* (Cambridge University Press, Cambridge, England, 1984).
[15] Y. Fujimoto, L. O'Raifeartaigh, and G. Parravicini, Nucl. Phys. **B212**, 268 (1983).
[16] B. Gato, L. León, J. Pérez-Mercader, and M. Quirós, Nucl. Phys. **B253**, 285 (1985).
[17] D. Hochberg, C. Molina-París, J. Pérez-Mercader, and M. Visser, hep-th/9809198.
[18] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[19] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (John Wiley, New York, 1989), Vol. II; see also Sect. 18.2 in Vol. II of Ref. [13].