Black Holes and Superconformal Mechanics

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The dynamics of a (super)particle near the horizon of an extreme Reissner-Nordström black hole is shown to be governed by an action that reduces to a (super)conformal mechanics model in the limit of large black hole mass.   [S0031-9007(98)07631-5]

A new class of interacting $(p + 1)$-dimensional conformal field theories has recently been discovered as the world-volume field theories on "test" p branes in the d-dimensional near-horizon background of other branes [1]. The key point is the fact that the near-horizon geometry is of the form $adS_{p+2} \times S^{d-p-2}$, with the adS isometries being realized on the test brane as conformal symmetry. Perhaps the simplest realization of this idea is provided by a charged point particle near the horizon of a $d = 4$ extreme Reissner-Nordström (RN) black hole. Here we use this example to elucidate some surprising connections between black holes and conformal invariance.

As an illustration of the issues, consider the conformal mechanics model of [2] (see also [3]) for the conjugate pair (p, x). The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{g}{2x^2}. \tag{1}$$

This was shown in [2] to a have a continuous spectrum of energy eigenstates with energy eigenvalue $E > 0$, but there is no ground state at $E = 0$. In the black hole interpretation of the model, the classical analog of an eigenstate of H is an orbit of a timelike Killing vector field k, equal to $\partial/\partial t$ in the region outside the horizon, and the energy is then the value of k^2. The absence of a ground state of H at $E = 0$ can now be interpreted as due to the fact that the orbit of k with $k^2 = 0$ is a null geodesic generator of the event horizon, which is not covered by the static coordinates adapted to ∂_t. The procedure used in [2] to cure this problem was to choose a different combination of conserved charges as the Hamiltonian. This corresponds to a different choice of time, one for which the world lines of static particles pass through the black hole horizon instead of remaining in the exterior spacetime.

Thus, the study of conformal quantum mechanics has potential applications to the quantum mechanics of black holes. Here we shall limit ourselves to an exposition of the classical aspects of this connection, and its supersymmetric extension. We start from the extreme RN metric in isotropic coordinates

$$ds^2 = -\left(1 + \frac{M}{\rho}\right)^{-2} dt^2 + \left(1 + \frac{M}{\rho}\right)^2$$
$$\times [d\rho^2 + \rho^2 d\Omega^2], \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the SO(3)-invariant metric on S^2, and M is the black hole mass, in units for which $G = 1$. The near-horizon geometry is therefore [4]

$$ds^2 = -\left(\frac{\rho}{M}\right)^2 dt^2 + \left(\frac{M}{\rho}\right)^2 d\rho^2 + M^2 d\Omega^2, \tag{3}$$

which is the Bertotti-Robinson (BR) metric [5]. It can be characterized as the $SO(1,2) \times SO(3)$ invariant conformally flat metric on $adS_2 \times S^2$. The parameter M may now be interpreted as the S^2 radius (which is also proportional to the radius of curvature of the adS_2 factor). A test particle in this near-horizon geometry provides a model of conformal mechanics in which the $SO(1,2)$ isometry of the background spacetime is realized as a one-dimensional conformal symmetry. If the particle's mass m equals the absolute value of its charge q, then this is just the $p = 0$ case of the construction of [1]. However, there is nothing to prevent us from considering $m \neq |q|$, and we begin by considering this more general case. We shall see that this leads to a new "relativistic" model of conformal mechanics. In the "nonrelativistic" limit, which can be viewed as a limit of large black hole mass, one recovers the Hamiltonian (1).

Various supersymmetric generalizations of conformal mechanics have been studied by Akulov and Pashnev and by Fubini and Rabinovici [6]. A relativistic generalization of one such model can be obtained from the radial dynamics of a superparticle in the near-horizon geometry of an extreme RN solution of $d = 4$ $N = 2$ supergravity. An important feature of the supersymmetric case is that the superparticle has a fermionic gauge invariance, "κ symmetry," when $m = |q|$. Since this reduces the total number of fermions by half, it leads to a considerable simplification of the Hamiltonian governing radial motion. To take advantage of this simplification, we shall consider here only the $m = |q|$ superparticle.

Conformal mechanics and black holes.—In horospherical coordinates $(t, \phi = \rho/M)$ for adS_2, the 4-metric and

Maxwell 1-form of the BR solution of Maxwell-Einstein theory are

$$ds^2 = -\phi^2 dt^2 + \frac{M^2}{\phi^2} d\phi^2 + M^2 d\Omega^2,$$

$$A = \phi \, dt.$$

$$(4)$$

The metric is singular at $\phi = 0$, but this is just a coordinate singularity and $\phi = 0$ is actually a nonsingular degenerate Killing horizon of the timelike Killing vector field $\partial/\partial t$. We now define a new radial coordinate r by

$$\phi = (2M/r)^2. \qquad (5)$$

The BR metric is then

$$ds^2 = -(2M/r)^4 dt^2 + (2M/r)^2 dr^2 + M^2 d\Omega^2. \quad (6)$$

Note that the Killing horizon in these coordinates is now at $r = \infty$.

The (static-gauge) Hamiltonian of a particle of mass m and charge q in this background is $H = -p_0$ where p_0 solves the mass-shell constraint $(p - qA)^2 + m^2 = 0$. This yields

$$H = (2M/r)^2 [\sqrt{m^2 + (r^2 p_r^2 + 4L^2)/4M^2} - q], \quad (7)$$

where $L^2 = p_\theta^2 + \sin^{-2}\theta \, p_\varphi^2$, which becomes minus the Laplacian upon quantization [with eigenvalues $\ell(\ell + 1)$ for integer ℓ]. We can rewrite this Hamiltonian as

$$H = \frac{p_r^2}{2f} + \frac{mg}{2r^2 f}, \qquad (8)$$

where

$$f = \frac{1}{2} [\sqrt{m^2 + (r^2 p_r^2 + 4L^2)/4M^2} + q], \quad (9)$$

and

$$g = 4M^2(m^2 - q^2)/m + 4L^2/m. \qquad (10)$$

This Hamiltonian defines a new model of conformal mechanics. The full set of generators of the conformal group are

$$H = \frac{1}{2f} p_r^2 + \frac{g}{2r^2 f}, \quad K = -\frac{1}{2} fr^2, \quad D = \frac{1}{2} rp_r,$$

$$(11)$$

where K generates conformal boosts (also called the generator of "special conformal" or "proper conformal" transformations) and D generates dilatations. It may be verified that the Poisson brackets of these generators close to the algebra of $Sl(2, R)$.

To make contact with previous work on this subject, we restrict to angular quantum number ℓ and consider the limit

$$M \to \infty, \qquad (m - q) \to 0, \qquad (12)$$

with $M^2(m - q)$ kept fixed. In this limit $f \to m$, so

$$H = \frac{p_r^2}{2m} + \frac{g}{2r^2}, \qquad (13)$$

with

$$g = 8M^2(m - q) + 4\ell(\ell + 1)/m. \qquad (14)$$

This is the conformal mechanics of [2,3]. For obvious reasons we shall refer to this as nonrelativistic conformal mechanics; the nonrelativistic limit can be thought of as a limit of large black hole mass. When $\ell = 0$ an "ultraextreme" $m < q$ particle corresponds to negative g and the particle falls to $r = 0$; i.e., it is repelled to $\phi = \infty$. On the other hand, a "subextreme" $m > q$ particle is pushed to $r = \infty$, which corresponds to it falling through the black hole horizon at $\phi = 0$. The force vanishes (again when $\ell = 0$) for an "extreme" $m = q$ particle, this being a reflection of the exact cancellation of gravitational attraction and electrostatic repulsion in this case. A static extreme particle of zero angular momentum follows an orbit of $\partial/\partial t$, and remains outside the black hole horizon.

Superconformal mechanics.—The nonrelativistic conformal mechanics described above was extended in [6] to an $SU(1, 1|1) \cong OSp(2|2)$ invariant superconformal mechanics. This can be truncated, for $g = 0$, to an $OSp(1|2)$ invariant superconformal mechanics, which we shall recover here as the nonrelativistic, a limit of a relativistic superconformal mechanics describing the radial motion of a *superparticle* with zero orbital angular momentum in the near-horizon geometry of the extreme RN solution of $d = 4$ $N = 2$ supergravity. It follows from the formula (10) that $g = 0$ for this model, since we assume both $m = |q|$ and $\ell = 0$. As will be shown elsewhere [7], the equation of motion of the $SU(1, 1|1)$-invariant superconformal mechanics with $g \neq 0$ is the nonrelativistic limit of the radial equation of a superparticle with nonzero angular momentum, but here we limit ourselves to the simpler case of $OSp(1|2)$ and zero angular momentum.

To define the superparticle action as an integral over the image w of the world line in superspace, we introduce (i) the superspace frame 1-forms $E^A = (E^a, E^{\alpha i})$ (where $\alpha = 1, 2$ is an $Sl(2, C)$ index and $i = 1, 2$ is an index of the $SU(2)_R$ R-symmetry group) and (ii) the superspace Maxwell 1-form A. The action may then be written as

$$S = -\int_w [m\sqrt{-g} - qA], \qquad (15)$$

where

$$g = E^a \otimes_s E^b \eta_{ab}. \qquad (16)$$

This action is obviously invariant (up to surface terms) under infinitesimal isometries of the background that leave invariant the Maxwell field strength 2-form $F = dA$, i.e., under transformations generated by vector superfields ξ for which

$$\mathcal{L}_\xi g = 0, \qquad \mathcal{L}_\xi F = 0. \qquad (17)$$

The algebra of (anti)commutators of the vector superfields ξ is, by definition, the algebra of the "isometry group of the background." In this case the isometry superalgebra is that of the supergroup $SU(1, 1|2)$ with bosonic subgroup

SU(1, 1) × SU(2). The SU(1, 1) × SU(2) subgroup is the isometry group of $adS_2 \times S^2$. This supergroup has eight real (four complex) supercharges as expected from the fact that the BR solution preserves all supersymmetries of $d = 4$ $N = 2$ supergravity. The anticommutator of these odd generators is [in SO(1, 2) × SO(3) notation]

$$\{Q_\alpha^i, \bar{Q}_j^\beta\} = -\frac{\delta_j^i}{4}(\hat{\gamma}^{\hat{m}\hat{n}})_\alpha^\beta \hat{M}_{\hat{m}\hat{n}} - \frac{\delta_\alpha^\beta}{4}(\gamma'^{m'n'})_j^i M'_{m'n'}. \quad (18)$$

The $\hat{\gamma}_{\hat{m}}$ generate the SO(1, 2) Clifford algebra and are chosen to be $\hat{\gamma}_0 = i\sigma_3$, $\hat{\gamma}_1 = \sigma_1$, and $\hat{\gamma}_2 = i\sigma_2$, where σ_i are the Pauli matrices. The $\gamma'_{m'}$ are the Pauli matrices generating the SO(3) Clifford algebra. $\bar{Q}_i^\alpha$ is the Dirac conjugate of Q_α^i in (1, 2) dimensions; i.e., $\bar{Q}_i^\alpha = i[(Q^i)^\dagger \hat{\gamma}_2 \hat{\gamma}_0]^\alpha$. The conformal SU(1, 1) charges (H, K, D) are packaged in $\hat{M}_{\hat{m}\hat{n}}$ as

$$H = -P_0 = -2(M_{02} + M_{01});$$
$$K = 2(M_{02} - M_{01}); \qquad D = 2M_{21}, \quad (19)$$

and $M'_{m'n'}$ are SO(3) generators.

We now define

$$\mathcal{Q}_\alpha = Q_\alpha^1 + \varepsilon_{\alpha\beta}\bar{Q}_1^\beta + Q_\alpha^2 + \varepsilon_{\alpha\beta}\bar{Q}_2^\beta, \quad (20)$$

and it follows that

$$\mathcal{Q}_\alpha = \begin{pmatrix} S \\ iQ \end{pmatrix}, \quad (21)$$

where Q and S are real. The anticommutator of these odd generators is

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = -M_{\alpha\beta}, \quad (22)$$

where

$$M_{\alpha\beta} = \begin{pmatrix} iK & D \\ D & iH \end{pmatrix}. \quad (23)$$

Thus the charges (H, K, D, Q, S) generate a sub-supergroup which is actually OSp(1|2; R) [the nonvanishing (anti)commutation relations are given in (31) below]. This is the sub-supergroup relevant to the truncated system in which we consider a superparticle moving radially. This system is equivalent to a $d = 2$ superparticle on a superspace with adS_2 "body" and isometry supergroup OSp(1|2; R), the Sp(2; R) ≅ SU(1, 1) subgroup being the isometry group of adS_2. This simplified model still captures the essential feature of the black hole, i.e., the existence of an event horizon.

One has only to gauge fix the reparametrization invariance of the action for a superparticle in this adS_2 superspace to find a model of superconformal mechanics, but unless $m = q$, both the standard supersymmetry and the conformal supersymmetry will be nonlinearly realized; i.e., there will be no state annihilated by either Q or S. This is hardly surprising since there is clearly no classical solution of zero energy when $g \neq 0$ whereas there is when $g = 0$. This distinction is reflected in the κ symmetry of the $m = |q|$ action which, for reasons explained in detail elsewhere, ensures that half of the supersymme-

tries are linearly realized. In the present context, it means that Q is linearly realized in that the ground state is annihilated by Q, while S is nonlinearly realized. This is the case that we are going to study in detail in this paper.

We proceed by first passing to the Hamiltonian form of the above superparticle action, which is a functional of the (2|2) superspace coordinate variables Z^M and their conjugate momenta p_M. The Lagrangian in this form is

$$L = \dot{Z}^M p_M - \frac{1}{2}v(\tilde{p}^2 + m^2) + \zeta^\alpha E_\alpha^M(p_M - qA_M), \quad (24)$$

where v is a Lagrange multiplier for the mass-shell constraint, ζ is a two-component real spinor Lagrange multiplier for the fermionic constraints, and

$$\tilde{p}_a = E_a^M(p_M - qA_M). \quad (25)$$

The fermionic constraints are purely second class if $m \neq q$, but half first class and half second class when $m = q$. Now, E_a^μ vanishes in flat superspace. It must therefore continue to vanish in any superconformally flat superspace since the supervielbeins are obtained in such cases from that of flat superspace by a super-Weyl transformation with scalar superfield parameter [8]. The BR background is superconformally flat, so we have

$$\tilde{p}^2 = g^{mn}(p_m - qA_m)(p_n - qA_n), \quad (26)$$

where $g^{mn} \equiv \eta^{ab}E_a^m E_b^n$. The mass-shell constraint for the superparticle is therefore formally identical to that of the bosonic particle. The only difference resides in the fact that the inverse metric g^{mn} and the Maxwell 1-form A_m are *superfields*. Their leading components are just the inverse metric and Maxwell 1-form of the bosonic action, but they will also contain terms proportional to fermions.

Now, all fermion terms in the expansion of g^{mn} and A_m must be even in fermions. In the special case that the superspace is (2|2) dimensional with adS_2 body, the expansion in fermions must terminate at the quadratic order because there are only two fermionic variables. If we further specialize to the $m = |q|$ case then only one combination of these two can actually appear (this is implied by κ invariance). Thus, all fermion bilinears vanish identically in this case and the mass-shell constraint, and hence the Hamiltonian is *identical* to that of the bosonic particle. The same is true of all the Sl(2; R) generators. The remaining generators of OSp(1|2; R) are the supersymmetry charge Q and the generator of superconformal boosts (alias "special" supersymmetry) S. These could be deduced from the charges associated with the fermionic Killing vector superfields of the background superspace, but it is easy to guess them as they are necessarily linear in the one physical fermion, which we may call ψ. The final result is as follows. The Sp(2; R) ≅ Sl(2; R) generators of this ($m = q$, $d = 2$) model are

$$H = \frac{1}{2f}p_r^2, \quad K = -\frac{1}{2}fr^2, \quad D = \frac{1}{2}rp_r, \quad (27)$$

where

$$f = \frac{1}{2} m[\sqrt{1 + (rp_r/2mM)^2} + 1], \qquad (28)$$

and the fermionic generators are

$$Q = \frac{p_r}{\sqrt{2f}} \psi, \qquad S = \sqrt{f/2}\, r\psi, \qquad (29)$$

where ψ is an anticommuting world line "field." Given the Poisson bracket (PB) relations

$$\{r, p_r\} = 1, \qquad \{\psi, \psi\} = i, \qquad (30)$$

one may verify that these generators define the Lie superalgebra of $OSp(1|2; R)$. Specifically, the nonzero PB relations are

$$\begin{aligned}
\{D, H\} &= H, & \{D, K\} &= -K, & \{H, K\} &= 2D, \\
\{D, Q\} &= \tfrac{Q}{2}, & \{D, S\} &= -\tfrac{S}{2}, & & \\
\{H, S\} &= -Q, & \{K, Q\} &= -S, & & \\
\{Q, Q\} &= iH, & \{S, S\} &= -iK, & \{Q, S\} &= iD.
\end{aligned} \qquad (31)$$

In the $M \to \infty$ limit we obtain an $OSp(1|2)$ invariant superconformal mechanics model with $g = 0$.

We have shown that the dynamics of a (super)particle in the near-horizon geometry of the extreme RN solution of $d = 4$ $N = 2$ supergravity is governed by a model of (super)conformal mechanics that generalizes previous constructions of such models. For purely radial motion, ($L^2 = 0$), and when $m = |q|$ there is a family of degenerate ground states of the particle Hamiltonian parametrized by $\langle r \rangle$. Because r scales under dilatations, conformal invariance is spontaneously broken for any finite or nonzero $\langle r \rangle$, but it is unbroken when either $\langle r \rangle = 0$ or $\langle r \rangle = \infty$. As explained in a slightly different context in [1], the quantity $\langle r \rangle / M$ is effectively the coupling constant, so the "end of the universe" limit $\langle r \rangle \to 0$ (recall that this corresponds to $\langle \phi \rangle \to \infty$) is equivalent to the $M \to \infty$ limit in which we obtain a free nonrelativistic superconformal mechanics. The other limit in which $\langle r \rangle \to \infty$ is an ultrarelativistic one in which the particle's orbit approaches a null geodesic generator of the Killing horizon. The Hamiltonian governing the particle's dynamics in this limit may be found by taking $M \to 0$ for fixed m and q. In the $L^2 = 0$ case this yields

$$H = \frac{2Mp_r}{r} + \mathcal{O}(M^2). \qquad (32)$$

By ignoring the $\mathcal{O}(M^2)$ terms, we effectively take the limit, and the $Sl(2, R)$ generators reduce to

$$H = \frac{2Mp_r}{r}, \quad K = -\frac{r^3 p_r}{8M}, \quad D = \frac{1}{2} rp_r. \qquad (33)$$

The M dependence may now be removed by the rescaling $r \to \sqrt{M}r$, $p_r \to p_r/\sqrt{M}$. The absence of any dependence of this Hamiltonian on m and q means that the full symmetry group of this model is that of the *massless* (super)particle in the same background. For superconformally flat backgrounds, such as adS_2 or the BR spacetime, the symmetry group is the same as that of a free particle in flat space, and is therefore an infinite rank extension of the superconformal group [9].

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