Excitation of Solitons by Adiabatic Multiresonant Forcing

L. Friedland* and A. G. Shagalov†
Racah Institute of Physics, Hebrew University of Jerusalem,
91904 Jerusalem, Israel
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It is shown that stable, large amplitude, spatially coherent solutions of the nonlinear Schrödinger equation can be excited by a weak forcing composed of an oscillation and a standing wave with a slowly varying frequency. The excitation involves autoresonant transition from a growing amplitude, uniform state to spatially modulated solution approaching the soliton, as the frequency increases in time. [S0031-9007(98)07585-1]

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The ac-driven, damped nonlinear Schrödinger equation (NLSE) $i\psi_t + \psi_{xx} + |\psi|^2\psi = -i\Gamma\psi + f$, where $f(t) = \varepsilon \exp(i\Lambda t)$ was proposed originally [1] as a model describing dipolar excitations in one-dimensional condensates. Later this equation was used in other applications, including ferromagnets in rotating magnetic fields [2], long Josephson junctions in ac fields [3], and rf-driven plasmas [4]. Among a variety of solutions of driven NLSE, the simplest are stationary, phase-locked states $\psi = a(x)\exp(i\Lambda t)$, where $a(x)$ satisfies $a_{xx} - \Lambda a + |a|^2 a = -i\Gamma a + \varepsilon$. The existence and stability of these solutions were addressed previously [1,5,6]. In the present work, we suggest a simple way of adiabatic excitation (from zero) of the phase-locked states by slowly varying a *single* parameter, i.e., by chirping the frequency of the forcing. The proposed scheme is based on imposing the periodic boundary condition $\psi(x,t) = \psi(x + L, t)$ and spatially modulating the amplitude of the driving force. In particular, we shall study the case $f(x,t) = \varepsilon(x)\exp[i\varphi(t)]$, where $\Lambda(t) = \varphi_t$ is a slow function of time, $\varepsilon(x) \equiv \varepsilon_0 + \varepsilon_1 \cos(k_0 x)$, and $k_0 = 2\pi/L$.

Our approach to controlling the nonlinear wave excitation process is based on the autoresonance effect, which reflects a natural tendency of the nonlinear system to preserve, under certain conditions, the resonance with external perturbations despite variation of the system's parameters. Applications of this idea to Korteweg–de Vries, sine-Gordon, and other nonlinear wave systems exist in the literature [7,8]. Recently, the autoresonance was also studied in the context of generating spatially modulated NLSE solutions [9]. However, the proposed scheme required a special choice of initial *and* boundary conditions, as well as space *and* time variation of parameters of the driver. Instead, in the present work, we formulate an initial value problem, vary the frequency of the driver *only*, and include a weak dissipation in our analysis.

We shall use vanishing initial conditions and view the driver as a perturbation. In this case, the initial evolution of our solution comprises a *linear* wave excitation problem. Since the driver is a combination of an oscillation and a standing wave with slowly varying frequencies, one expects this linear excitation to be effective at times when one of the driver components passes the resonance with a linear NLSE wave. The linear dispersion relation of the dissipationless NLSE is $\omega + k^2 = 0$, and, therefore, the resonances with ε_0 or ε_1 components of the driver are expected when $\Lambda \approx 0$ or $\Lambda \approx -k_0^2$, respectively. Suppose Λ is increasing in time and one passes $\Lambda = 0$ resonance first (this is the scenario of our excitation scheme). We illustrate such a case in Fig. 1, showing $|\psi|$ found numerically by a standard spectral method [10]. The driving frequency was $\Lambda(t) = d + \Lambda_0 \sin[(\pi t)/(2T_0)]$ for $|t| \le T_0$ and $\Lambda = d + \Lambda_0$, for $t > T_0$, and we used $T_0 = 300$, $d = 3$, $\Lambda_0 = 4.5$, $L = \pi$, $\varepsilon_{0,1} = 0.05$. Finally, we neglected damping in this example and applied zero initial conditions (at $t_0 = -T_0$). Since, initially, $0 > \Lambda > -k_0^2$, the system passed the linear resonance $\Lambda = 0$ first (at $t = t_1 \approx -140$). One can see in the figure that a quasi-uniform (flat) NLSE solution is excited in the vicinity of this linear resonance. The solution grows at later times, but remains flat until, at $t = t_2 \approx -30$, it develops a spatially modulated profile and, beyond $t > 300$, assumes an almost solitory form. Additional results from the same calculations are presented in Fig. 2, showing the evolution of the maximum value of $|\psi|$ over $0 \le x \le L$ and the phase mismatch $\Delta = [\arg(\psi) - \varphi(t)] \bmod 2\pi$ at $x = L/2$. The

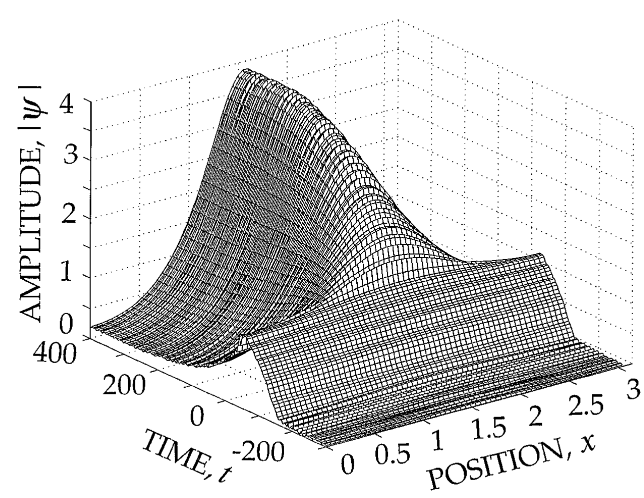

FIG. 1. Two-stage autoresonant excitation of a solitory wave.

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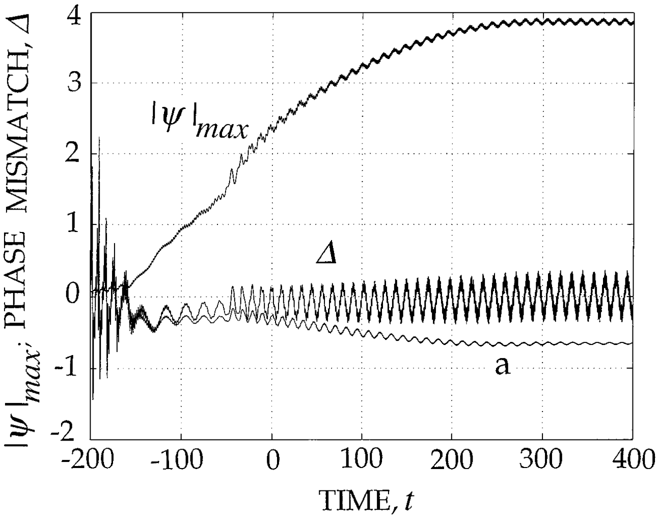

FIG. 2. Evolution of $|\psi|_{\max}$ and Δ (at $x = L/2$). Curve a shows the effect of dissipation.

curve a in the same figure represents Δ obtained by adding a small dissipation, $\Gamma = 0.005$ (the smoothed slow evolution of $|\psi|_{\max}$ was the same with and without dissipation). One can see that, beyond $t = t_1$, the phases of the solutions are locked to that of the driver, and both $|\psi|$ and Δ oscillate around slowly varying averaged values. Finally, in Fig. 3, we show a part of the evolution ($-45 < t < 5$) in the same example as the dissipative case in Fig. 2, but with $\varepsilon_1 = 0$, i.e., when the driver is a purely temporal oscillation. One can see the portion of the flat solution, as in Fig. 1. Nevertheless, $|\psi|$ does not exhibit *stationary* spatial profile beyond t_2, but transforms into a breather-like solution. The following theory explains these results.

We proceed by discussing the flat, phase-locked parts of the solutions in Figs. 1–3. By writing $\psi = A(x,t) \times \exp\{i[\varphi(t) + \Phi(x,t)]\}$, $\mathrm{Im}(A,\Phi) = 0$, NLSE yields

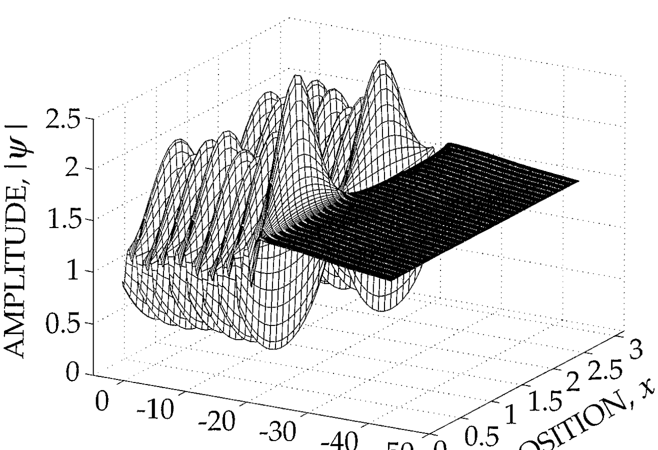

FIG. 3. Evolution of $|\psi|$ for purely temporal forcing $f = \varepsilon_0 \exp(i \int \Lambda\, dt)$. The autoresonance is destroyed at $t \approx -30$ due to the modulational instability.

$$A_t + A\Phi_{xx} + 2A_x\Phi_x = -\varepsilon \sin\Phi - \Gamma A\,,$$
$$A\Phi_t - A_{xx} + A\Phi_x^2 + \Lambda A - A^3 = -\varepsilon\cos\Phi\,. \quad (1)$$

Assuming an *almost* flat solution of (1) and using the periodicity condition, we expand A and Φ in the Fourier series and leave only the spatially independent parts and small first spatial harmonics in the expansions: $A \approx A_0(t) + \mathrm{Re}[A_1(t)\exp(ik_0x)]$ $\Phi \approx \Phi_0(t) + \mathrm{Re}[\Phi_1(t) \times \exp(ik_0x)]$, $|A_1| \ll A_0$, $|\Phi_1| \ll 1$. Then by linearizing

$$A_{0t} = -\Gamma A_0 - \varepsilon_0 \sin\Phi_0\,,$$
$$\Phi_{0t} = A_0^2 - \Lambda - (\varepsilon_0/A_0)\cos\Phi_0\,, \quad (2)$$

and

$$A_{1t} - A_0k_0^2\Phi_1 = -\Gamma A_1 - \varepsilon_0\Phi_1\cos\Phi_0 - \varepsilon_1\sin\Phi_0\,,$$
$$A_0\Phi_{1t} + (\Phi_{0t} + k_0^2 + \Lambda - 3A_0^2)A_1 \quad (3)$$
$$= \varepsilon_0\Phi_1\sin\Phi_0 - \varepsilon_1\cos\Phi_0\,.$$

By setting $I \equiv A_0^2$, $\theta \equiv \Phi_0$, Eqs. (2) can be written as $I_t = -H_\theta - 2\Gamma I$; $\theta_t = H_1$, where $H(I,\theta) \equiv \frac{1}{2}I^2 - \Lambda(t)I - 2\varepsilon_0 I^{1/2}\cos\theta$. Note that $H(I,\theta)$ has the characteristic form of *single resonance* Hamiltonian (expressed in terms of the action-angle variables) in a nonlinear resonance problem [11]. It is known that, by varying $\Lambda(t)$, this dynamical problem may exhibit temporal autoresonance [12], provided one starts from a sufficiently small I and passes the linear resonance $\Lambda = 0$ in time. In autoresonance, the angle θ ($= \Phi_0$) remains locked near $\theta = 0 (\mathrm{mod}\, 2\pi)$, while the action of the oscillator (and therefore, A_0) self-adjusts to preserve the approximate *nonlinear* resonance condition $A_0^2 = I \approx \Lambda(t)$ despite the variation of Λ. Also, both I and θ perform slow autoresonant oscillations around the smooth averages with characteristic frequency of $O(\varepsilon^{1/2})$. This dynamical analog explains the excitation of the flat, phase-locked solutions of NLSE seen in Figs. 1–3 at times $t_1 < t < t_2 \approx -30$.

The evolution in all our examples enters a new stage when the driving frequency passes (at $t = t_2$) the value $\Lambda = \frac{1}{2}k_0^2$, i.e., when the amplitude of the flat state becomes $A_0 \approx \Lambda^{1/2} = 2^{-1/2}k_0$. In analyzing this effect, we must discuss the behavior of the spatial modulations A_1 and Φ_1 in the developed stage ($A_0k_0^2 \gg \varepsilon$) of the flat autoresonant solution. These modulations are described by Eq. (3), where one views A_0 and Φ_0 as known. To lowest order in ε, we use $A_0^2 \approx \Lambda(t)$ and $\Phi_0 \approx 0$ in (3), yielding

$$A_{1t} - \Lambda^{1/2}k_0^2\Phi_1 = -\Gamma A_1\,,$$
$$\Lambda^{1/2}\Phi_{1t} + (k_0^2 - 2\Lambda)A_1 = -\varepsilon_1\,. \quad (4)$$

Next, we define the *slow* quasisteady state

$$\overline{A}_1 = \varepsilon_1(2\Lambda - k_0^2)^{-1},$$
$$\overline{\Phi}_1 = \Gamma\varepsilon_1[\Lambda^{1/2}k_0^2(2\Lambda - k_0^2)]^{-1}, \quad (5)$$

and write the solutions of (4) in the form $A_1 = \overline{A}_1 + \delta A_1$, $\Phi_1 = \overline{\Phi}_1 + \delta\Phi_1$, where δA_1 and $\delta\Phi_1$ are assumed to be small and evolving as $\exp(-i\int \nu\, dt)$. Then, by neglecting the slow time dependence of the coefficients in (4), one obtains local characteristic equation $\nu^2 + i\Gamma\nu = k_0^2(k_0^2 - 2\Lambda)$. The latter can be recognized as describing the modulational instability of the flat solution [13]. In the $\Gamma = 0$ case, the instability condition is $k_0^2 - 2\Lambda < 0$, i.e., for increasing Λ, one expects a modulationally unstable behavior beyond $t = t_2$, where $\Lambda(t_2) = \frac{1}{2}k_0^2$. Nevertheless, we see the onset and evolution of this instability in Fig. 3 only. In this case $\varepsilon_1 = 0$ and $\overline{A}_1 = \overline{\Phi}_1 = 0$ during the excitation of the flat state [see Eq. (5)]. Therefore, starting $t = t_2$, the instability develops from numerical noise (or small initial conditions, if applied). Beyond t_2, the resonance is destroyed and a new damped, breather-like state is observed. In contrast, if $\varepsilon_1 \neq 0$, Eq. (5) predicts a significant increase of $\overline{A}_1$ (and $\overline{\Phi}_1$ in the dissipative case) as one approaches the instability threshold ($\Lambda \to \frac{1}{2}k_0^2$), and one may expect violation of our linearized analysis and a different *nonlinear* evolution, as seen in Figs. 1 and 2. We must use a *fully* nonlinear treatment of the driven *modulated* wave in analyzing this case.

Our new physical picture is based on interpreting the solution of (1) at any given time as being a *slightly* perturbed solution of the same system of equations, but with the time derivatives, the forcing term, and the dissipation set to zero. In other words, we write $A \approx r$, $\Phi \approx \phi$, where r and ϕ satisfy

$$\begin{aligned} r\phi_{xx} + 2r_x\phi_x &= 0\,, \\ r_{xx} - r\phi_x^2 - \Lambda(t)r + r^3 &= 0\,, \end{aligned} \tag{6}$$

with $\Lambda(t)$ fixed at a given time. Equation (6) describes a two degrees of freedom Hamiltonian problem, where one interprets x as "time," while r and ϕ are the polar coordinates of a quasiparticle moving in the central force potential well $V_{\rm eff}(r) = -\frac{1}{2}\Lambda r^2 + \frac{1}{4}r^4$. This problem is integrable, since the angular momentum $M \equiv r^2\phi_x$ and the energy $E \equiv \frac{1}{2}(r_x^2 + M^2/r^2) + V_{\rm eff}(r)$ are conserved (in x). We focus on oscillating solutions of the problem and introduce the canonical action-angle variables (J, Θ) and (M, ξ), where the first pair describes purely "radial" oscillations [14] of the quasiparticle, while the second is associated with the azimuthal motion. Since the pair of actions $\{J, M\}$ is conserved, one can use it or, alternatively, the pair $\{E, M\}$ for labeling the solutions of (6). Then, the desired solutions of system (1) are approximated by $A(x,t) \approx r[E(t), M(t), \Theta(x,t)]$ and $\Phi(x,t) \approx \phi[E(t), M(t), \Theta(x,t), \xi(x,t)]$, where the "energy" E and the "angular momentum" M are slowly varying in time. The goal is to find the temporal evolution of these labeling parameters. We shall use our recent theory [9] in solving this problem, but, first, give a qualitative description of the results in Fig. 1 in terms of our dynamical analog.

We view the evolution in Fig. 1 as reflecting a continuous, *slow* (in real time) change of almost "radial" ($M \approx 0$) oscillatory motion of the quasiparticle in the central force potential $V_{\rm eff}(r, \Lambda)$. We notice that the driving terms $\varepsilon\cos\Phi$ and $\varepsilon\sin\Phi$ in (1), if added on the right-hand sides of our dynamical system (6), can be interpreted as representing an additional "uniform" driving force of strength $\varepsilon = \varepsilon_0 + \varepsilon_1\cos(k_0x)$ and pointing in the $\phi = 0$ direction in the "plane" of motion. The resonance of the quasiparticle with this force plays an important role as the potential $V_{\rm eff}$ evolves in time. We illustrate this evolution in Fig. 4, showing $V_{\rm eff}$ for different values of $\Lambda(t)$. Curve a in the figure corresponds to a negative value $\Lambda = -1.5$. Initially, at $t = t_0$, the quasiparticle is located at the bottom of this potential well and has zero energy E and angular momentum M. A simple calculation shows that for $\Lambda < 0$, the natural wave number (recall, x plays the role of "time" in the dynamical problem) of small "radial" oscillations of the quasiparticle at the bottom of the well is $2|\Lambda|$. Then, since $k_0 > 2|\Lambda|$ in the initial excitation stage ($t_0 < t < t_1$), the component $\varepsilon_1\cos(k_0x)$ of the driving force is out of resonance. Consequently, the actions J and M are nearly conserved and the particle remains at the bottom ($J \approx 0$) of the potential well (the open circle on curve a). As the result, the amplitude $A \approx r$ remains small for $\Lambda < 0$. At $t = t_1$, Λ becomes positive. This affects the form of the potential $V_{\rm eff}$, and the new form is represented by curves b and c ($\Lambda = 0.8, 2$) in Fig. 4. Now the minimum of the potential is negative and moves to increasing radii, $r = r_0 = \Lambda^{1/2}(t)$. At the same time, the natural "wave number" of small "radial" oscillations at the bottom of the potential well is again 2Λ. Therefore, the driving force $\varepsilon_1\cos(k_0x)$ is still nonresonant, as long as $2\Lambda < k_0^2$, ($t_1 < t < t_2$). The particle stays at the bottom of the potential well ($J \approx 0$), and, consequently, moves to larger radii r_0 (the open circles on curves b and c). Since $A \approx r_0$, this stage corresponds to a growing amplitude, flat (x-independent) solution of NLSE. Note that the constant component ε_0 of the driving force slightly tilts the quasipotential in the $\phi = 0$ direction, braking the

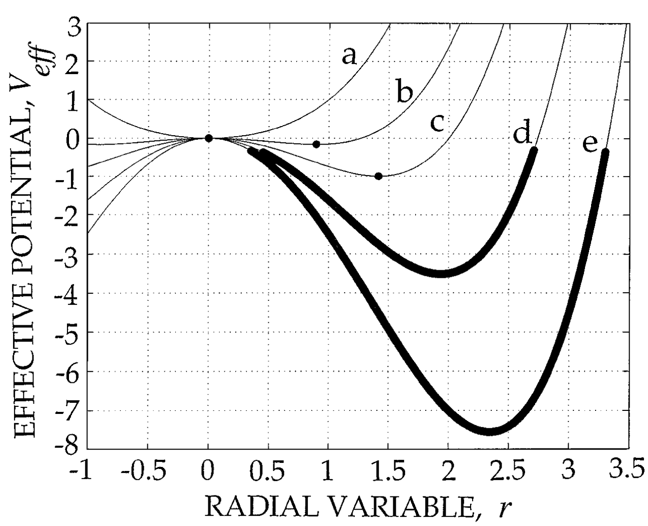

FIG. 4. Potential $V_{\rm eff}(r, \Lambda)$ in the dynamical analog. (a) $\Lambda < 0$; the quasiparticle is at the bottom of the well. (b,c) $0 < \Lambda < \frac{1}{2}k_0^2$. The particle is still at the boom of the well. (d,e) $\Lambda > \frac{1}{2}k_0^2$, "radial" oscillations, conserving $K \approx k_0$.

central symmetry, and forcing the quasiparticle to remain at the azimuthal angle $\phi = 0$ continuously, as it moves to larger "radii." Finally, as the frequency Λ passes (at $t = t_2$) the value of $\frac{1}{2}k_0^2$, the wave number k_0 of the driver crosses the resonance with the natural wave number of small "radial" oscillations near the bottom of V_{eff} and one efficiently excites these oscillations. Beyond the linear resonance, the system enters the *spatial autoresonance* regime, i.e., the state when, for $t > t_2$, the nonlinear wave number $K \equiv \partial E(J,M)/\partial J$ *remains* nearly the same as that of the driving modulation (k_0), despite the variation of the system's parameters. Since K depends on $\Lambda(t)$ parametrically, the labeling parameters E and/or M must vary in time in order to satisfy the autoresonance condition $K(E,M,\Lambda) \approx k_0$. One finds (see below) that M remains small, so only the "energy" E departs from the minimum value in the potential well and the dynamical system develops "radial" oscillations of increasing amplitude [the motion shown by thick solid lines on the potentials d ($\Lambda = 3.75$) and e ($\Lambda = 5.5$) in Fig. 4]. The corresponding NLSE solution assumes a phase-locked, spatially modulated form. Finally, as one further increases the driving frequency Λ, the system approaches the separatrix of the "radial" oscillations, meaning the emergence of the soliton solution. This completes the qualitative picture of the excitation process seen in Fig. 1. Next, we explain the observed stability of this process.

We shall neglect the dissipation, for simplicity, in studying the stability issue. Then, the problem reduces to a particular case of a more general stability theory [9] for driven, phase-locked, standing wave solutions of NLSE. We describe our temporal evolution problem via Whitham's [15] averaged variational principle $\delta \int \mathcal{L}\, dt = 0$, where $\mathcal{L}$ is the Lagrangian of the original driven problem, averaged over the fast angular variable (the canonical angle Θ of the "radial" oscillations of the quasiparticle in our application). This Lagrangian depends on slow variables and parameters only; i.e., $\mathcal{L} = \mathcal{L}[E, M, \xi, \mu; \Lambda(t)]$, where $\mu \equiv \Theta - k_0 x$ is the phase mismatch of the spatial oscillations. When $\mathcal{L}$ is known, the variations with respect to E, M, ξ, and μ yield evolution equations for the slow independent variables. We shall use $\mathcal{L}$ derived in [9], neglect the time variation of Λ in studying the stability, assume the existence of a phase-locked steady state $\overline{E}$, $\overline{\mu} = 0$, $\overline{M} = 0$, $\overline{\xi} = 0$, and add perturbations δE, δM, $\delta \mu$, and $\delta \xi$. Then linearized variational equations for the perturbed variables are [16]

$$\begin{aligned} &\overline{K}_E \delta E + \overline{K}_\Lambda \delta \xi_t = 0\,, \\ &\delta M_t - (2k_0)^{-1}\langle r^2 \rangle \delta \mu_{tt} = 2k_0 \varepsilon_1 \overline{\alpha}_1 \delta \mu\,, \\ &\langle r^{-2} \rangle \delta M - (2k_0)^{-1} \delta \mu_t = 0\,, \\ &k_0(\overline{J}_{\Lambda E} \delta E_t + \overline{J}_{\Lambda\Lambda} \delta \xi_{tt}) = -(\varepsilon_0 \overline{\alpha}_0 - \varepsilon_1 \overline{\alpha}_1) \delta \xi\,, \end{aligned} \tag{7}$$

where $J(E,\Lambda)$ is the action of *purely* "radial" ($M = 0$) oscillations, $\langle \cdots \rangle$ denotes the averaging over Θ, $\overline{(\cdots)}$ represents evaluation at $E = \overline{E}$ and $M = 0$, while $\overline{\alpha}_{0,1}$ are the zero and first harmonic coefficients (both real) in the Fourier expansion of $r = r(\Theta)$. System (7) yields $\delta \mu_{tt} = -g^\mu \delta \mu$ and $\delta \xi_{tt} = -g^\xi \delta \xi$, where $g^\mu \equiv \varepsilon_1 \alpha_1 4k_0^2 [\langle r^2 \rangle - \langle r^{-2} \rangle^{-1}]^{-1}$ and $k_0 g^\xi \equiv (\varepsilon_0 \overline{\alpha}_0 - \varepsilon_1 \overline{\alpha}_1)(\overline{J}_{\Lambda\Lambda} - \overline{J}_{\Lambda E} \overline{K}_\Lambda / \overline{K}_E)^{-1}$. For stability, both g^ξ and g^μ must be positive. One finds that g^μ is always positive, but the positiveness of g^ξ requires $\varepsilon_0 \overline{\alpha}_0 - \varepsilon_1 \overline{\alpha}_1 > 0$ throughout the excitation process, requiring a sufficiently large value of ε_0. This is the necessary condition for having a continuous double phase locking, $\xi \approx 0$ and $\mu \approx 0$, in our system. Our calculations also show that sufficiently weak dissipation in the problem does not destroy the autoresonance.

In conclusion, we have described the excitation and control of phase-locked, spatially modulated solutions of NLSE by using *two-component* autoresonant forcing. We have presented a simple dynamical analog associated with the problem and studied the stability of the proposed excitation scheme via averaged variational principle. It seems promising to apply similar ideas to driven systems for which NLSE is a small amplitude approximation. Experimental observation of autoresonantly excited nonlinear waves also comprises a challenging goal for future studies.

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*Email address: lazar@vms.huji.ac.il
†Present address: Institute of Metal Physics, Ekaterinburg 620219, Rusian Federation.

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