Bohr's Correspondence Principle and the Area Spectrum of Quantum Black Holes

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During the last twenty-five years evidence has been mounting that a black-hole surface area has a *discrete* spectrum. Moreover, it is widely believed that area eigenvalues are *uniformly* spaced. There is, however, no general agreement on the *spacing* of the levels. In this Letter we use Bohr's correspondence principle to provide this missing link. We conclude that the area spacing of a black hole is $4\hbar \ln 3$. This is the unique spacing consistent both with the area-entropy *thermodynamic* relation for black holes, with the Boltzmann-Einstein formula in *statistical physics*, and with *Bohr's correspondence principle*. [S0031-9007(98)07615-7]

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The necessity in a quantum theory of gravity was already recognized in the 1930s. However, despite the flurry of activity on this subject we still lack a complete theory of quantum gravity. It is believed that black holes may play a major role in our attempts to shed some light on the nature of a quantum theory of gravity (such as the role played by atoms in the early development of quantum mechanics).

The quantization of black holes was proposed long ago in the pioneering work of Bekenstein [1]. The idea was based on the remarkable observation that the horizon area of nonextremal black holes behaves as a classical *adiabatic invariant*. In the spirit of the Ehrenfest principle [2], any classical adiabatic invariant corresponds to a quantum entity with *discrete* spectrum, Bekenstein conjectured that the horizon area of a nonextremal quantum black hole should have a discrete eigenvalue spectrum.

To elucidate the *spacing* of the area levels it is instructive to use a semiclassical version of Christodoulou's reversible processes. Christodoulou [3] showed that the assimilation of a neutral (*point*) particle by a (nonextremal) black hole is reversible if it is injected at the *horizon* from a radial *turning point* of its motion. In this case the black-hole surface area is left unchanged and the changes in the other black-hole parameters (mass, charge, and angular momentum) can be undone by another suitable (reversible) process. (This result was later generalized by Christodoulou and Ruffini for charged point particles [4].)

However, as was pointed out by Bekenstein in his seminal work [5], the limit of a *point* particle is not a legal one in *quantum* theory. In other words, the particle cannot be both at the horizon and at a turning point of its motion; this contradicts the *Heisenberg quantum uncertainty principle*. As a concession to quantum theory Bekenstein ascribes to the particle a *finite* proper radius b while continuing to assume, in the spirit of Ehrenfest's theorem, that the particle's center of mass follows a classical trajectory. Bekenstein [5] has shown that the assimilation of a finite size neutral particle inevitably causes an increase in the horizon area. This increase is minimized if the particle is captured when its center of mass is at a turning point a proper distance b away from the horizon [5]:

$$(\Delta A)_{\min} = 8\pi\mu b\,, \tag{1}$$

where A is the black-hole surface area and μ is the rest mass of the particle. For a point particle $b = 0$ and one finds $\Delta A_{\min} = 0$. This is Christodoulou's result for a reversible process. However, a quantum particle is subjected to quantum uncertainty. A relativistic quantum particle cannot be localized to better than its Compton wavelength. Thus, b can be no smaller than $\hbar/\mu$. This yields a lower bound on the increase in the black-hole surface area due to the assimilation of a (neutral) test particle

$$(\Delta A)_{\min} = 8\pi l_p^2, \tag{2}$$

where $l_p = (\frac{G}{c^3})^{1/2}\hbar^{1/2}$ is the Planck length (we use gravitational units in which $G = c = 1$). It is easy to check that the reversible processes of Christodoulou and Ruffini and the lower bound Eq. (2) of Bekenstein are valid only for *non*extremal black holes. Thus, for nonextremal black holes there is a *universal* (i.e., independent of the black-hole parameters) minimum area increase as soon as one introduces quantum nuances to the problem.

The universal lower bound Eq. (2) derived by Bekenstein is valid only for *neutral* particles [5]. Recently, Hod [6] analyzed the capture of a quantum (finite size) *charged* particle by a black hole and found a similar lower bound. The lower bound on the area increase caused by the assimilation of a charged particle is given by [6]

$$(\Delta A)_{\min} = 4l_p^2. \tag{3}$$

As was noted by Bekenstein [5] (for neutral particles) the underlying physics which excludes a completely reversible process is the *Heisenberg quantum uncertainty principle*. However, for *charged* particles it must be supplemented by another physical mechanism [6]—a Schwinger discharge of the black hole (*vacuum polarization* effects). Without this physical mechanism one could have reached the reversible limit.

It is remarkable that the lower bound found for charged particles is of the same order of magnitude as the one given by Bekenstein for neutral particles, even though they emerge from *different* physical mechanisms. The *universality* of the fundamental lower bound (i.e., its independence on the black-hole parameters) is clearly a strong evidence in favor of a *uniformly* spaced area spectrum for quantum black holes (see Ref. [7]). Hence, one concludes that the quantization condition of the black-hole surface area should be of the form

$$A_n = \gamma l_p^2 n; \qquad n = 1, 2, \ldots, \tag{4}$$

where γ is a dimensionless constant.

It should be recognized that the precise values of the universal lower bounds Eqs. (2) and (3) can be challenged. These lower bounds follow from the assumption that the smallest possible radius of a particle is precisely equal to its Compton wavelength. Actually, the particle's size is not so sharply defined. Nevertheless, it should be clear that the fundamental lower bound must be of the same order of magnitude as the one given by Eq. (3); i.e., we must have $\gamma = O(4)$. The small uncertainty in the value of γ is the price we must pay for not giving our problem a full quantum treatment. In fact, the analyses presented in Refs. [5,6] are analogous to the well known semiclassical determination of a lower bound on the ground state energy of the hydrogen atom [2]. Both analyses consider a *classical* object (an electron or a test particle) subjected to the Heisenberg uncertainty principle. The analogy with usual quantum physics suggests the next step—a *wave* analysis of black-hole perturbations.

The evolution of small perturbations of a black hole are governed by a one-dimensional wave equation. This equation was first derived by Regge and Wheeler for perturbations of the Schwarzschild black hole [8]. Furthermore, it was noted that, at late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [9–11]. To describe these free oscillations of the black hole the notion of quasinormal modes was introduced [12]. The quasinormal mode frequencies (ringing frequencies) are characteristic of the black hole itself.

The perturbation fields outside the black hole are governed by a one-dimensional Schrödinger-like wave equation (assuming a time dependence of the form e^{-iwt}):

$$\frac{d^2\Psi}{dr_*^2} + [w^2 - V(r)]\Psi = 0, \tag{5}$$

where the tortoise radial coordinate r_* is related to the spatial radius r by $dr_* = dr/(1 - \frac{2M}{r})$ and the effective potential is given by

$$V(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{\sigma}{r^3}\right], \tag{6}$$

where M is the black-hole mass, l is the multipole moment index, and $\sigma = 2, 0, -6$ for scalar, electromagnetic, and gravitational perturbations, respectively.

The black hole's free oscillations (quasinormal modes) correspond to solutions of the wave equation (5) with the physical boundary conditions of purely outgoing waves at spatial infinity ($r_* \to \infty$) and purely ingoing waves crossing the event horizon ($r_* \to -\infty$) [13]. The quasinormal modes are related to the pole singularities of the scattering amplitude in the black-hole background. The ringing frequencies are located in the complex frequency plane characterized by $\mathrm{Im}(w) < 0$. It turns out that for a given l there exist an infinite number of quasinormal modes for $n = 0, 1, 2, \ldots$ characterizing modes with decreasing relaxation times (increasing imaginary part) [14,15]. On the other hand, the real part of the frequency approaches a constant value as n is increased.

Our analysis is based on *Bohr's correspondence principle* (1923): "transition frequencies at large quantum numbers should equal classical oscillation frequencies." Hence, we are interested in the asymptotic behavior (i.e., the $n \to \infty$ limit) of the ringing frequencies. These are the highly damped black-hole oscillations frequencies, which are compatible with the statement (see, for example, [16]) "quantum transitions do not take time" (let $w = w_R - iw_I$, then $\tau \equiv w_I^{-1}$ is the effective relaxation time for the black hole to return to a quiescent state. Hence, the relaxation time τ is arbitrarily small as $n \to \infty$.)

The determination of the highly damped quasinormal mode frequencies of a black hole is not a simple task. This is a direct consequence of an exponential divergence of the quasinormal mode eigenfunctions at $r_* \to \infty$. In fact, the asymptotic behavior of the ringing frequencies is known only for the simplest case of a Schwarzschild black hole. Nollert [17] found that the asymptotic behavior of the ringing frequencies of a Schwarzschild black hole is given by

$$Mw_n = 0.043\,712\,3 - \frac{i}{4}\left(n + \frac{1}{2}\right) + O[(n+1)^{-1/2}]. \tag{7}$$

It is important to note that the highly damped ringing frequencies depends only upon the black-hole mass and is *independent* of l and σ. This is a crucial feature, which is consistent with the interpretation of the highly damped ringing frequencies (in the $n \gg 1$ limit) as characteristics of the *black hole* itself. The asymptotic behavior Eq. (7) was later verified by Andersson [18] using an independent analysis.

We note that the numerical limit $\mathrm{Re}(w_n) \to 0.043\,712\,3M^{-1}$ (as $n \to \infty$) agrees (to the available data given in [17]) with the expression $\ln 3/(8\pi)$. This identification is supported by thermodynamic and statistical physics arguments discussed below. Using the relations $A = 16\pi M^2$ and $dM = E = \hbar w$ one finds $\Delta A = 4l_p^2 \ln 3$. Thus, we conclude that the dimensionless constant γ appearing in Eq. (4) is $\gamma = 4 \ln 3$ and the area spectrum for the quantum Schwarzschild black hole is

given by

$$A_n = (4l_p^2 \ln 3)n; \qquad n = 1, 2, \ldots . \tag{8}$$

This result is remarkable from a *statistical physics* point of view. The semiclassical versions of Christodoulou's reversible processes Refs. [5,6], which naturally lead to the conjectured area spectrum Eq. (4), are at the level of mechanics, not statistical physics. In other words, these arguments did not relay in any way on the well known thermodynamic relation between black-hole surface area and entropy. In the spirit of the Boltzmann-Einstein formula in statistical physics, Mukhanov and Bekenstein [7,16,19] relate $g_n \equiv \exp[S_{\rm BH}(n)]$ to the number of microstates of the black hole that correspond to a particular external macrostate ($S_{\rm BH}$ being the black-hole entropy). In other words, g_n is the degeneracy of the nth area eigenvalue. The accepted thermodynamic relation between black-hole surface area and entropy [5] can be met with the requirement that g_n has to be an integer for every n only when

$$\gamma = 4 \ln k; \qquad k = 1, 2, \ldots . \tag{9}$$

Thus, statistical physics arguments force the dimensionless constant γ in Eq. (4) to be of the form Eq. (9). Still, a specific value of k requires further input, which was not available so far. This Letter provides a first independent derivation of the value of k. It should be mentioned that following the pioneering work of Bekenstein [1] a number of independent calculations (most of them in the last few years) have recovered the uniformly spaced area spectrum Eq. (4) [20–27]. However, there is no general agreement on the spacing of the levels. Moreover, *none* of these calculations is compatible with the relation $\gamma = 4 \ln k$, which is a direct consequence of the accepted thermodynamic relation between black-hole surface area and entropy. The relation $\gamma = 4 \ln 3$ derived in this Letter is the *only* one consistent both with the area-entropy thermodynamic relation, with statistical physics arguments, and with Bohr's correspondence principle.

The universality of black-hole entropy (i.e., its direct thermodynamic relation to black-hole surface area) and the universality of the lower bounds Eqs. (2) and (3) (i.e., their independence of the black-hole parameters) suggest that the area spectrum Eq. (8) should be valid for a generic Kerr-Newman black hole. Moreover, our analysis leads to a natural conjecture on the asymptotic behavior of the highly damped quasinormal modes of a generic Kerr-Newman black hole. Using the first law of black-hole thermodynamics,

$$dM = \Theta dA + \Omega dJ , \tag{10}$$

where $\Theta = \frac{1}{4}(r_+ - r_-)/A$ and $\Omega = 4\pi a/A$ [$r_\pm = M \pm (M^2 - a^2 - Q^2)^{1/2}$ are the black hole's (event and inner) horizons and $a = J/M$ is the black-hole angular momentum per unit mass], one finds

$$\mathrm{Re}(w_n) \to 4\Theta \ln 3 + \Omega m , \tag{11}$$

as $n \to \infty$, where m is the azimuthal eigenvalue of the field. The asymptotic behavior of the Kerr-Newman ringing frequencies was not determined directly so far. This is a direct consequence of the numerical complexity of the problem. It is of great interest to compare the conjectured asymptotic behavior given in this Letter with the results of direct numerical computations.

In summary, using a semiclassical version of Christodoulou's reversible processes (see Refs. [5,6]) one can derive a fundamental lower bound on the increase in black-hole surface area. The *universality* of the fundamental lower bound (i.e., its independence of the black-hole parameters) is a strong evidence in favor of a *uniformly* spaced area spectrum for quantum black holes. However, the spacing between area eigenvalues cannot be determined to better than an order of magnitude (the results presented in Ref. [6] suggest that the area spacing is of order $4l_p^2$). This is a direct consequence of the semiclassical nature of these analyses. An analogy with usual quantum physics suggests the next step—a *wave* analysis of black-hole perturbations. Applying Bohr's correspondence principle to the ringing frequencies which characterize a black hole, we derive the missing link. We find the area spacing to be $4l_p^2 \ln 3$, which is in excellent agreement with the value predicted by the semiclassical analysis [6]. Moreover, this result is remarkable from a *statistical physics* point of view. The area spacing $4l_p^2 \ln 3$ derived in this Letter is the *unique* value consistent both with the area-entropy *thermodynamic* relation, with *statistical physics* arguments (namely, with the Boltzmann-Einstein formula), and with *Bohr's correspondence principle*.

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