

## Fractals at $T = T_c$ due to Instantonlike Configurations

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We investigate the geometry of the critical fluctuations for a general system undergoing a thermal second order phase transition. Adopting a generalized effective action for the local description of the fluctuations of the order parameter at the critical point ( $T = T_c$ ) we show that instantonlike configurations, corresponding to the minima of the effective action functional, build up clusters with fractal geometry characterizing locally the critical fluctuations. The connection between the corresponding (local) fractal dimension and the critical exponents is derived. Possible extension of the local geometry of the system to a global picture is also discussed. [S0031-9007(98)07547-4]

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In a system undergoing a second order phase transition, coherent fluctuations at all scales occur at the critical point. The understanding of the geometry of these fluctuations is a long-standing problem [1]. The self-similarity of the critical fluctuations suggests the formation of clusters, with a nonvanishing value of the order parameter, which have a fractal structure [2,3]. Simple geometrical arguments lead to the conclusion that the fractal dimension, describing the critical clusters, reflects the scaling properties of the underlying fluctuations and therefore can be related to the critical exponents characterizing the phase transition [3]. Rigorous mathematical derivation and a deeper understanding of the origin of such a relation is, however, missing in the general case. Some important efforts have been performed concerning the geometry of the critical clusters in the Ising and Potts models [4] where the order parameter is described through a discrete field variable. For the more general case of continuous fields there is, however, to our knowledge, no such understanding. It is the purpose of the present paper to illuminate the way the geometry of the critical clusters emerges in the case of a continuous effective field theory. For our considerations we study a self-interacting scalar field  $\phi$  at thermal equilibrium. The effective action of the thermal system, at the critical point  $T = T_c$  of the continuous phase transition, can be specified, in a wide class of critical phenomena, by an effective theory in  $d$  dimensions, in terms of a macroscopic field  $\phi$  (order parameter) as follows:

$$\Gamma_c[\phi] = g_1 \Lambda^{-d-2} \int d^d x \times \left[ \frac{1}{2} (\nabla_d \phi)^2 + g_2 \Lambda^{2d+2} |\Lambda^{-d} \phi|^{\delta+1} \right]. \quad (1)$$

The dimension of the field  $\phi$  in (1) has been chosen:  $\phi \sim (\text{volume})^{-1}$  and the ultraviolet cutoff  $\Lambda$  of the underlying microscopic theory fixes the coarse graining scale  $R_c \approx \Lambda^{-1}$  of the effective system (throughout this Letter we use the convention  $\kappa_B = 1$  (Boltzman constant) and the energy

is given in inverse length units). The form (1) for constant fields leads to the standard equation of state at  $T = T_c$ :  $\frac{\delta \Gamma_c}{\delta \phi} \sim \phi^\delta$  ( $\phi > 0$ ) and therefore the index  $\delta$  is identified with the isothermal critical exponent of the system. For a high-temperature phase transition, the coarse graining cutoff  $\Lambda$  is bounded by the critical temperature itself ( $\Lambda \gg T_c$ ) and the dimensionless parameters  $g_1, g_2$  in Eq. (1) are expressed in terms of the ratio  $\lambda = \frac{\Lambda}{T_c}$  ( $\lambda \gg 1$ ). In fact, using as a concrete example the  $O(N)$  3D effective theory, the action  $\Gamma_c[\phi]$  in the large  $N$  limit and for a fixed orientation in the internal  $O(N)$  space, is written as follows [5]:

$$\Gamma_c[\phi] = \lambda^5 \Lambda^{-5} \int d^3 x \times \left[ \frac{1}{2} (\nabla \phi)^2 + 2 \left( \frac{2\pi \lambda^5}{N} \right)^2 \Lambda^8 (\Lambda^{-3} \phi)^6 \right].$$

It belongs to the general class (1) with  $d = 3$ ,  $\delta = 5$ ,  $g_1 = \lambda^5$ ,  $g_2 = 2 \left( \frac{2\pi \lambda^5}{N} \right)^2$  and to the particular sector  $g_1 \gg 1$ .

The scalar field  $\phi$  in the following will be no further specified. It can describe magnetization density or particle density or the density of any other extensive physical quantity characterizing the phase transition. Introducing the dimensionless quantities  $\hat{\phi} = \Lambda^{-d} \phi$ ,  $\hat{x}_i = \Lambda x_i$ , we can rewrite the effective action (1) as

$$\Gamma_c[\hat{\phi}] = g_1 \int_V d^d \hat{x} \left[ \frac{1}{2} (\nabla \hat{\phi})^2 + g_2 |\hat{\phi}|^{\delta+1} \right]. \quad (2)$$

In what follows we use, for simplicity, the old notation  $(\phi, x_i)$  instead of  $(\hat{\phi}, \hat{x}_i)$ .

The statistics of the critical system is resolved if we are able to calculate the partition function:

$$Z = \int \mathcal{D}[\phi] e^{-\Gamma_c[\phi]}. \quad (3)$$

The nontrivial task is to carry out the path integration in (3). Since the self-similarity is a subtle symmetry of the system and it is expected to dominate in the formation of the

fractal geometry of the clusters (the precise definition of the cluster will be given later on), the conventional methods to perform the path integration, which can be found in the literature [6], are not suitable for the problem at hand. We propose therefore to perform the summation, with an appropriate measure, over a class of saddle-point configurations which are expected to dominate the critical fluctuations in Eq. (2) for  $g_1 \gg 1$ . In order to illustrate our method we will treat, for simplicity, the one-dimensional case in some detail. Our approach can be, however, extended to higher dimensions without difficulties.

Let us now become more quantitative. In the one-dimensional case the partition function of the critical system ( $T = T_c$ ) is given as

$$Z = \int \mathcal{D}[\phi] \times \exp \left\{ -g_1 \int_0^R dx \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + g_2 |\phi|^{\delta+1} \right] \right\}, \quad (4)$$

where  $R$  is the size of the considered system. We assume here that our investigations refer to an open subsystem of the entire physical system located in the vicinity of the point  $x = 0$ . Therefore no restrictions to the values of the order parameter at the boundaries of the subsystem are imposed. The geometrical properties of the subsystem  $R$  are expressed through the scaling properties of the extensive quantities characterizing the subsystem with varying size  $R$  around  $x = 0$ .

The saddle-point configurations  $\phi(x)$  fulfill the Euler-Lagrange equation corresponding to the effective action  $\Gamma_c[\phi]$  and describe the classical motion in the concave potential  $U(\phi) \sim -|\phi|^{\delta+1}$ . This equation can be solved analytically in terms of two parameters  $E$  and  $\phi(0)$ , where  $E$  is a conserved (during the classical motion) quantity identified with the total energy of the moving particle:

$$E = \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 - g_2 |\phi|^{\delta+1}. \quad (5)$$

Using Eq. (5) one can show that configurations with  $E \neq 0$  contribute to the partition function  $Z$  with a suppression factor  $e^{-g_1 R |E|}$  suggesting that the dominant saddle points contributing to (4) come from those solutions of the equations of motion for which  $E \approx 0$ . In fact Eq. (5) can be integrated to give, for  $E = 0$ , instantonlike solutions of

the form

$$\phi(x) = \left( \frac{c}{\sqrt{2g_2}} \right)^{2/(\delta-1)} \times \left[ \frac{c}{\sqrt{2g_2}} \phi(0)^{-(\delta-1)/2} \pm x \right]^{-[2/(\delta-1)]}; \quad (\delta > 1) \quad (6)$$

with  $c = \frac{2}{\delta-1}$ . Setting  $x_o = \mp \frac{c}{\sqrt{2g_2}} \phi(0)^{-(\delta-1)/2}$  the  $\phi$  field for  $E = 0$  simplifies to

$$\phi(x) = A |x - x_o|^{-[2/(\delta-1)]}, \quad (7)$$

$$A = \left[ \frac{g_2}{2} (\delta - 1)^2 \right]^{-[1/(\delta-1)]}.$$

To perform the path integration in (4) we have to sum up the contributions of all instantonlike saddle-point configurations of the form (7), i.e., to integrate over the parameter  $x_o$ . In order to determine the correct integration measure we consider the class of solutions (7) with  $x_o \gg R$ . In this case  $\phi(x) = \text{const} \sim x_o^{-[2/(\delta-1)]}$  and in a region of radius  $R$  around  $x = 0$  the path integration becomes an ordinary integral over  $x_o$  with measure  $\mathcal{D}\phi = d\mu(x_o) \approx x_o^{-(\delta+1)/(\delta-1)} dx_o$ . To determine the range of integration over  $x_o$  we have first to clarify the meaning of a cluster in our picture: Let us assume that  $M$  is an extensive variable (i.e., magnetization) characterizing the field configurations of the critical system and possessing a minimal, in general, different from zero, value  $\mu$  ( $M \geq \mu$ ) related to the microscopic details of the system. Within the picture of the local observer positioned at  $x = 0$  a cluster of size  $R$  is the set  $S$  of points with a maximum distance  $R$  from the origin. The most appropriate observable to study the geometric properties of the critical clusters is the thermal average  $\langle M(R) \rangle = \langle \int_0^R \phi(x) dx \rangle$  and in particular its behavior as a function of  $R$ . The minimum value  $\mu$  of the magnetization  $M$  introduces a threshold  $\phi_{\min}$  to the configurations  $\phi$  contributing to this average, leading us to an upper limit for the integral  $x_o$  as a function of  $R$ :  $x_o \leq \left( \frac{AR}{\mu} \right)^{(\delta-1)/2}$ . On the other hand the singularity at  $x = x_o$  must lie outside the region of the considered cluster  $(0, R)$  restricting the integration in (4) over configurations with  $x_o \geq R$ . In terms of the partition function (4) this average can be determined as

$$\left\langle \int_0^R \phi(x) dx \right\rangle = \frac{A^{(\delta-1)/(\delta-3)}}{Z} \int_R^{(AR/\mu)^{(\delta-1)/2}} dx_o x_o^{-(\delta+1)/(\delta-1)} [x_o^{(\delta-3)/(\delta-1)} - (x_o - R)^{(\delta-3)/(\delta-1)}] \times \exp \left\{ -G_1 \left( \frac{\delta-1}{\delta+3} \right) [(x_o - R)^{-(\delta+3)/(\delta-1)} - x_o^{-(\delta+3)/(\delta-1)}] \right\} \quad (8)$$

with  $G_1 = 2g_1 g_2 A^{\delta+1}$ . One can show analytically that in the large  $G_1$  limit ( $G_1 \gg 1$ ) there are three characteristic regions determining the behavior of the integral in Eq. (8). Putting  $R_d = A^{-(\delta+1)/\delta} \mu^{(\delta+1)/\delta} G_1^{1/\delta}$  and  $R_u = G_1^{(\delta-1)/(\delta+3)}$  we find that in the central region of scales  $R_d \ll R \ll R_u$ :  $\langle \int_0^R \phi(x) dx \rangle \sim R^{\delta/(\delta+1)}$  leading to a fractal structure of the cluster around the point  $x = 0$  with a fractal mass dimension [7]:  $d_f = \delta/(\delta+1)$ . This behavior crosses over for

$R \gg R_u$  to a different power law:  $\langle \int_0^R \phi(x) dx \rangle \sim R^{(\delta-3)/(\delta-1)}$  suggesting the presence of a fractal with mass dimension  $\tilde{d}_F = \frac{\delta-3}{\delta-1}$  at large scales.

For  $R \ll R_d$  a violation of the scaling symmetry of the critical cluster is revealed leading to an approximately constant value of the integral (8). The parameter  $R_d$  defines a minimal scale of the critical system effectively related with the minimal value  $\mu$  of the order parameter. In Fig. 1a we show the numerical results for the calculation of (8) using the values  $G_1 = 5 \times 10^8$  and  $\delta = 5$ . The three different regions and the corresponding crossover scales describing the geometry of the critical cluster are clearly distinguished. In the same plot we show also the corresponding linear fits to illustrate more transparently the above considerations. The fractality in the central region characterizes the critical system in the sense that it corresponds to the scaling behavior in the vicinity of the local observer when  $\mu \rightarrow 0$ . This is at best shown in Fig. 1b where we calculated (8) using the same value of  $G_1$  as in Fig. 1a and let the upper limit in the  $x_o$  integration going to infinity. The crossover scale  $R_u$  gives presumably a measure of the correlation length of the finite system at  $T = T_c$ .

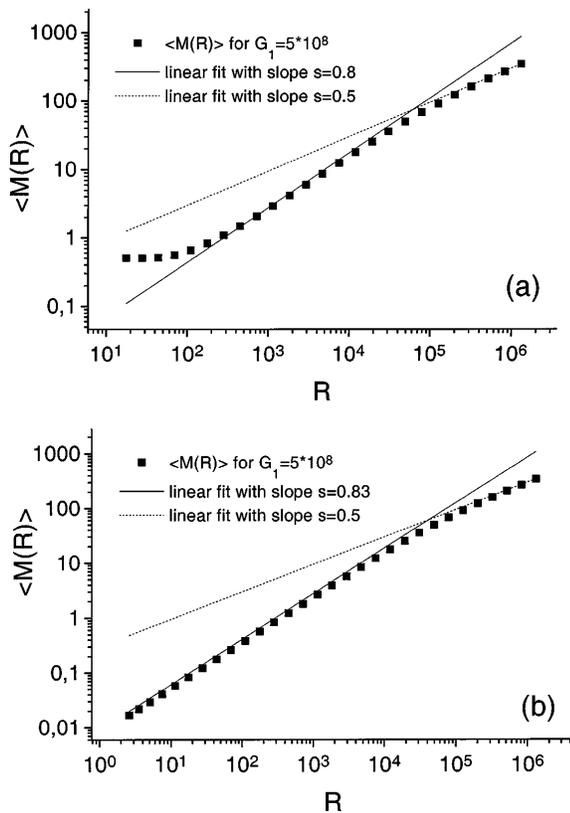


FIG. 1. (a) The mean magnetization  $\langle M(R) \rangle$  as a function of  $R$  (in units  $\Lambda^{-1}$ ) for the  $d = 1$  case,  $G_1 = 5 \times 10^8$ , and  $A = 1$ . The fitted lines indicate the two regions of fractality as described in the text. A nonvanishing minimum magnetization  $\mu$  takes care for the violation of the scaling in small distances  $R$ . (b) The mean magnetization  $\langle M(R) \rangle$  for  $d = 1$ ,  $G_1 = 5 \times 10^8$ , and  $A = 1$  for  $\mu \rightarrow 0$ .

One can now easily generalize the above investigations in order to describe systems of higher dimensions ( $d > 1$ ). We proceed in a similar way as for the one-dimensional case using the saddle point approximation for the partition function  $Z_d$ :

$$Z_d = \int \mathcal{D}[\phi] e^{-g_1 \int_v d^d x [(1/2)(\nabla_d \phi)^2 + g_2 |\phi|^{\delta+1}]} \quad (9)$$

in order to calculate the thermal average  $\langle \int_0^R \phi(\vec{x}) d^d x \rangle_{Z_d}$ . The summation over the saddle points in Eq. (9) becomes an ordinary integration over the position of the singularity in the instantonlike solutions  $\phi_d$  in  $d$  dimensions. In an analogous way as in the 1D case, solving the Euler-Lagrange equations for the critical action, we get instantonlike solutions for  $\phi_d$ . In the case  $d = 2$  a class of analytic solutions possessing a point singularity can be determined:

$$d = 2; \quad \phi_2(\vec{r}) = A_2 |\vec{r} - \vec{r}_o|^{-2/(\delta-1)}; \\ A_2 = \left[ \frac{g_2}{2} (\delta - 1)^2 \right]^{-1/(\delta-1)}. \quad (10)$$

Performing the calculation of the mean value  $\langle M(R) \rangle = \langle \int d^2 \vec{r} \phi(\vec{r}) \rangle$ , characterizing a two-dimensional critical cluster, in an analogous way as for the 1D case, we get a similar behavior concerning its fractal geometric properties. There are characteristic scales in the radial component  $R_d = A_2^{-(\delta+1)/2\delta} \mu^{(\delta+1)/2\delta} G_2^{1/2\delta}$  and  $R_u = G_2^{(\delta-1)/4}$  with  $G_2 = \pi g_1 \left[ \frac{2A_2^2}{(\delta-1)^2} + g_2 A_2^{\delta+1} \right]$  such that

$$d = 2; \quad \langle M(R) \rangle \sim R^{2\delta/(\delta+1)}; \quad R_d \ll R \ll R_u, \\ \langle M(R) \rangle \sim R^{2(\delta-2)/(\delta-1)}; \quad R_u \ll R. \quad (11)$$

A crossover for large  $R$  is found also in this case. For dimensions  $d \geq 3$  no analytic solution to the Euler-Lagrange equations is available in the general case of a nonvanishing anomalous dimension  $\eta$  ( $\delta = \frac{d+2-\eta}{d-2+\eta}$ , [8]). However, these equations can be integrated numerically leading again to an instantonlike behavior. In particular, one can find exact analytic spherical solutions of this kind for  $d \geq 3$  in the special case when  $\eta = 0$  ( $\delta = \frac{d+2}{d-2}$ ).

For  $0 < \eta \ll 1$  an approximate solution can be obtained given as follows:

$$d \geq 3; \quad \phi_d(r) = A_d (r_o^2 - r^2)^{(2-d)/2}; \\ A_d = \left( \frac{(d-2)r_o}{\sqrt{2g_2}} \right)^{(d-2)/2} \left( \frac{(d-2)}{\sqrt{2g_2} r_o} \right)^{d\eta/4}. \quad (12)$$

The solution (12) goes to the exact one for  $\eta = 0$ . For  $r$  far from the singularity region the approximate form (12) coincides practically with the exact (numerically obtained) solution. This can be at best seen in Fig. 2a where we plot together the numerical and the approximate solution to the Euler-Lagrange equations for  $d = 3$  and  $\eta = 0.34$ . In fact, in a wide range of universality classes including the  $O(4)$  theory in which  $\eta \approx 0.034$  [9], the anomalous dimension for  $d = 3$  is much smaller [10] and therefore

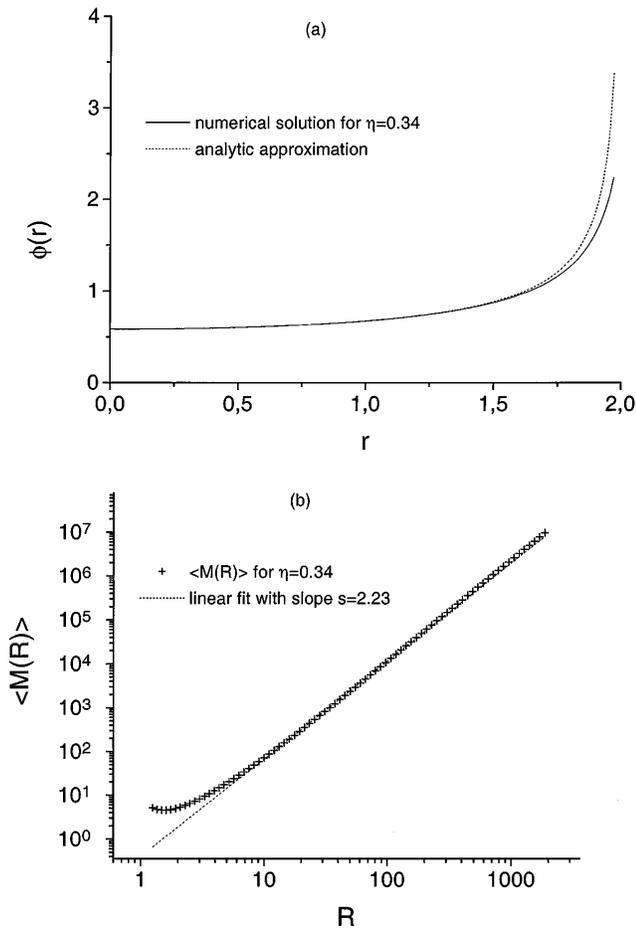


FIG. 2. (a) A saddle-point solution to the action (1) for  $d = 3$  and  $\eta = 0.34$ . The analytical approximation (dashed line) and the result of the numerical integration (solid line) are displayed separately. (b) The mean magnetization  $\langle M(R) \rangle$  for  $d = 3$  and  $G_3 = 10^2$ . The fitted line indicates the fractality in the central region.

one can safely use the solution (12) for most calculations. Based on (12) we determined  $\langle M(R) \rangle$  for spherically symmetric clusters in  $d \geq 3$  dimensions and explored the geometric properties of such a cluster. Once again we got the typical central fractality region crossing over to a fractal with a smaller dimension for distances comparable to the correlation length. It must be noted that for  $d \geq 3$  the crossover disappears as  $\eta \rightarrow 0$ . Using  $R_d = a^{-(\delta+1)/d\delta} \times \mu^{(\delta+1)/d\delta} (G_d)^{1/d\delta}$  and  $R_u = (G_d)^{-1/[d+q(\delta+1)]}$  with  $a = \left(\frac{d-2}{\sqrt{2g_2}}\right)^{[d-2+(d\eta/2)]/2}$ ,  $G_d = \frac{2a^{\delta+1} \pi^{d/2}}{d\Gamma(d/2)} g_1 g_2$ , and  $q = \frac{2-d-(d\eta/2)}{2}$ , we obtain

$$d \geq 3; \quad \langle M_d(R) \rangle \sim R^{1+[(d-\eta)/2]}; \quad R_d \ll R \ll R_u, \\ \langle M_d(R) \rangle \sim R^{1+[d(2-\eta)/4]}; \quad R_u \ll R. \quad (13)$$

The characteristic behavior of  $\langle M(R) \rangle$  for  $d = 3$  is presented in Fig. 2b. Here we used  $G_3 = 10^2$  and  $\eta = 0.34$  (as in Fig. 2a). The breaking of the fractality (for  $R \ll R_d$ ) is clearly reproduced while the crossover is suppressed due to the small value of  $\eta$ . The power laws  $\langle M_d(R) \rangle \sim R^{d_F}$  or  $\sim R^{\tilde{d}_F}$  with  $d = 1, 2, \dots$  determine fractals at different scales with dimensions  $d_F, \tilde{d}_F$ . Putting together our re-

sults in one, two, and three dimensions and taking into account that our considerations for  $d \geq 3$  are restricted to the case when the anomalous dimension  $\eta$  is small ( $\eta \ll 1$ ) we can cast our results for the fractal properties of the critical cluster into universal expressions determining  $d_F$  and  $\tilde{d}_F$  in terms of  $d$  and  $\delta$ :

$$d_F = \frac{d\delta}{\delta + 1}; \quad \tilde{d}_F = d - \frac{2}{\delta - 1}; \\ d_F - \tilde{d}_F = \frac{\eta(d - 2 + \eta)}{2(2 - \eta)}. \quad (14)$$

The expression for  $d_F$  is in accordance with the results obtained in [3,4] for the Ising and Potts critical clusters. Thus we have found that Eqs. (14) describe the fractal geometry of the critical clusters in a wide range of scales and for a general class of effective theories. Considering the asymptotic region where the size of the cluster reaches the value of the correlation length of the finite system we find a crossover to a more dilute phase with a smaller fractal dimension  $\tilde{d}_F$ . Furthermore we have revealed a mechanism responsible for the formation of the geometry of the critical clusters. Our considerations are restricted to the point of view of a local description. The generalization of our approach in order to build up the entire critical system requires the extension of the formalism to configurations incorporating many, suitably located, instantonlike structures (of the size of the correlation length) covering the whole available space. Such an investigation, however, goes beyond the scope of the present Letter.

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