Can Stochastic Resonance Lead to Order in Chaos?

Hideki Yamazaki
Fukuoka Institute of Technology, Department of Electronics, 3-30-1 Wajirohigashi, Fukuoka 811-0295 Japan

Tomoji Yamada
Kyushu Institute of Technology, Department of Electronics, 1-1 Sensui-cho, Tobata-ku, Kitakyushu 804-0015 Japan

Shoichi Kai
Kyushu University, Department of Applied Physics, 6-10-1 Hakozaki, Fukuoka 812-8581 Japan
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The dynamical behavior of two coupled chaotic oscillators was experimentally studied under superposition of both noise and periodic external forces. Increasing the noise intensity the chaotic behavior was suppressed and the signal-to-noise ratio (SNR) became maximum at a certain noise intensity; that is, the stochastic resonance phenomenon was observed. In order to define the states more quantitatively and compare to the change of SNR the largest Lyapunov exponent was calculated which showed a drastic change from a positive value in the chaotic state to zero in the resonance regimes; noise-induced order was observed. [S0031-9007(98)07632-7]

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In general, noise has been thought to be an obstacle. Much effort has been expended to suppress the generation and influence of noise. However, the usefulness of noise has recently been recognized [1,2], for example, the noise-induced detection of subthreshold signals or the noise-induced transmission of information by sensory neurons, i.e., the so-called stochastic resonance (SR) phenomenon. Theoretical works on conventional (classical) SR have well explained the improvement of the signal-to-noise ratio (SNR), as well as features of the power spectral density of the output, with increasing noise intensity [2–5].

The SR phenomena in electric circuits systems were reported by several researchers [6,7] after a first demonstration using a Schmitt trigger circuit [8], and then the more adequately theoretical description was given for the nonmonotonic behavior of SNR observed in these systems [2,5]. In addition to these demonstrations, SR was found in various devices and applications: a ring laser [9], analog simulators [6,10], an electron paramagnetic resonance system [11], a superconducting loop with a Josephson junction [12], an actual nerve cell [13], and a neural network in mammalian brain [14]. All these studies focus the main attention on the behavior of classical SR in either bistable or monostable systems (see, for example, a review paper Ref. [2]).

Very recently the SR(-like) phenomena were investigated for chaotic systems and spatially extended systems by means of computer simulations, e.g., noise effect in the Lorenz attractor with periodic signals [15], chaotic stochastic resonance in the FitzHugh-Nagumo model [16], and SR near bifurcations [17] and SR in on-off intermittency in a randomly controlled logistic map [18]. For a spatially extended system, an interesting result was reported that the minimization of the largest Lyapunov exponent was observed at optimal noise intensity [19].

On the other hand, independently of SR, noise influences on chaotic attractors were reported, for example, transition to chaos in periodic windows induced by noise in the Lorenz model [20]. An especially interesting phenomenon has been reported that a more regular motion in the chaotic one could be obtained by the application of noise [21,22]. This is the so-called noise-induced order (NIO), which was first found, e.g., in the computer simulation done by Matsumoto and Tsuda [21].

In all the past investigations for SR in chaotic motions, thus, either SNR or Lyapunov exponents was discussed as well as power spectra and no relationship between SNR and the Lyapunov exponents has been reported so far. Since both SR and NIO are noise-induced periodic phenomena, however, their underlying mechanisms may resemble each other and may be discussed in a similar way. One may thus expect to observe NIO in coupled chaotic oscillator systems [23,24] by superimposing noise.

The questions to be investigated in the present study are as follows: (1) Can SR lead to order in two coupled chaotic oscillators? (2) What kind of mechanism can induce SR in such systems? (3) Is traditional noise-induced order the same phenomenon as SR?

Experiments were performed using two coupled oscillators, each of which consisted of a tunnel diode (1S1763), a resistor, a sinusoidal signal, and a noise source (Fig. 1). A Gaussian white noise generator (NF WG-722) with two independent and uncorrelated noises was used, for each of which the bandwidth was 0–50 kHz. The noise intensity Q_n was defined as the root-mean-square voltage $\sqrt{\langle V_n(t)^2 \rangle}$. A synthesized signal generator (YOKOGAWA FG120) was used as a periodic signal source.

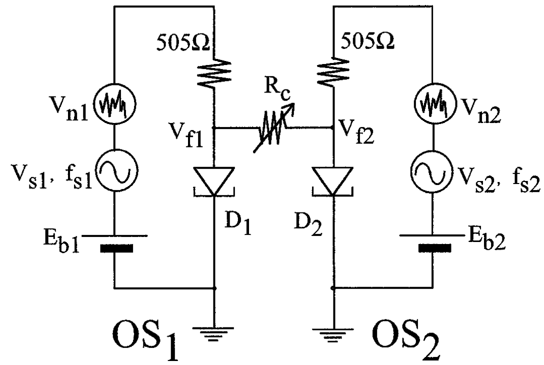

FIG. 1. Electronic circuits used in our experiment.

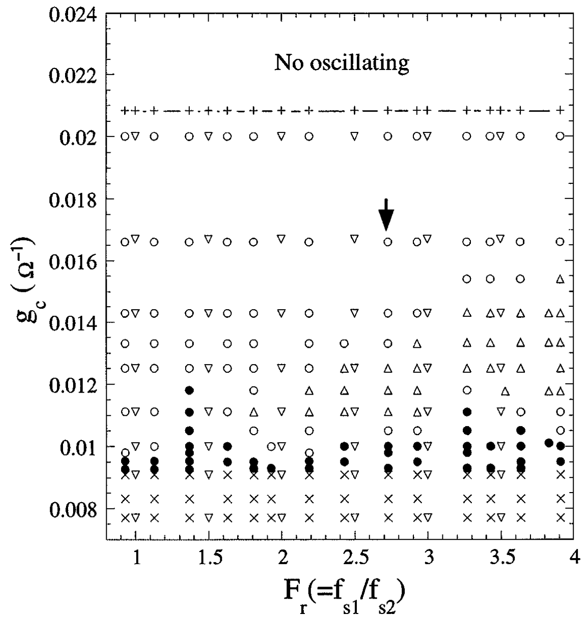

FIG. 2. State diagram of present circuits shown in Fig. 1. $E_{b1} = 18.85$ mV, $E_{b2} = 19.99$ mV, $V_{s1} = 3.57$ V, and $V_{s2} = 1.50$ V. The symbols $+$, $\bigtriangledown$, $\bigtriangleup$, $\times$, $\bigcirc$, and $\bullet$ denote no oscillation states, periodic-1 states, periodic-2 states, quasiperiodic-1 states, quasiperiodic-2 states, and chaotic states, respectively.

Output signals V_{f1} and V_{f2} were analyzed using a personal computer and a fast Fourier transform (FFT) analyzer (ADVANTEST R9211C), both of which are voltages across the tunnel diodes D_1 and D_2, respectively. The bias voltages E_{b1} and E_{b2} were adjusted so that the diodes were operating in a bistable state [25]. The switching time of a tunnel diode between the two states was quite short (about 35 nsec), and therefore the wave form of the output signals had a clear rectangular shape. We, respectively, call this transferring between two states and the interval time between two rectangular waves "the switching" and "the switching interval" hereafter. We determined the dynamical state of the circuits using wave forms of the output signals, time sequences of switching intervals, and their power spectral densities, as well as the Lyapunov exponent.

We calculated here the so-called largest Lyapunov exponent λ from the output signal using the method proposed by Wolf *et al.* in order to determine the qualitative aspect of the transition from chaos to order [26]. We finally obtained the delay time τ_1 and the evolution time τ_2 as $\tau_1 = \tau_2 = 1/f_{s_1}$ searching the best condition. Then the embedding dimension was determined as eight dimensions, for which the sufficient convergency of the Lyapunov exponent was guaranteed. We have also chosen as scalmax = 7.5% and scalmin = 0.5%. The influence of variation of these values on calculated results was much smaller than that of τ_1 and τ_2.

As shown in Fig. 2, we first determined the state diagram of the coupled circuits from λ as well as their return maps and power spectrum in the absence of noise as changing applied frequencies (f_{s1} and f_{s2}) and coupling strength g_c ($= 1/R_c\Omega^{-1}$). The bias voltages E_{b1} and E_{b2} across the tunnel diodes were 18.85 and 19.98 mV, respectively, where the state of each uncoupled single oscillator was in a bistable state (i.e., no spontaneous oscillation) in the absence of external signal [25]. The amplitudes of external signals V_{s1} and V_{s2} were 3.57 and 1.50 V, respectively. All data points are shown in Fig. 2 marked by symbols such as down and up triangles, crosses, and open and closed circles, respectively, indicating periodic 1, 2, quasiperiodic 1, 2, and chaotic states. Since the measuring points are discrete, we cannot describe the behavior between these points. However, general aspects of the state diagram are follows.

For the case of a simple rational number of the frequency ratio $F_r = f_{s1}/f_{s2}$ such as $F_r = 1.0$, 1.5, 2.0, 2.5, 3.0, and 3.5, only periodic and no oscillation states (large g_c) were observed. The periodic states in these regimes were characterized by either period-one or period-two type in return maps. No chaotic state was observed. For the case of an irrational number of F_r, five states such as no oscillation, quasiperiodic 1 and 2, periodic, and chaotic states were observed depending on g_c. Two different quasiperiodic states were possibly distinguishable, i.e., quasiperiodic 1 (QP1) and 2 (QP2) states. In the QP1 state, the return map formed a small circle and several sharp peaks were observed in its power spectrum. In the QP2 state, several dots appeared in the return map and mainly several sharp peaks were observed in its power spectrum. In the periodic state, the return map was the period-3 type and a few peaks were observed in its power spectrum. In the chaotic state, the complicated return map and the broad band spectrum were observed.

Thus after careful determination of the state diagram and of the desired conditions to study, we superimposed noises on periodic signals $f_{s1} = 273.973$ Hz and $f_{s2} = 100.000$ Hz. In these initial conditions, however, OS_1 was entrained to the external force while OS_2 located at one stable state and did not oscillate when the two oscillators were uncoupled. Under these conditions, as g_c increased, the following dynamical states were observed marked by the arrow in Fig. 2: At $g_c < 0.0091$ QP1 states were realized. For $0.0091 < g_c < 0.01$ chaotic states were observed. For $0.0111 < g_c < 0.0125$ and $0.0125 < g_c < 0.0208$ periodic 2 states and QP2 states were, respectively, observed. Finally, for $g_c > 0.0208$ no oscillation was observed.

In the chaotic state (noise-free and $g_c = 0.009\,75$), both oscillated irregularly and desynchronized. Figure 3(a) shows the power spectral density of a chaotic output signal. Many peaks are observed, and the background noise is much higher than that in the case of superimposing noise [Fig. 3(b)]. As the largest Lyapunov exponent was positive in this case, the irregular motion observed is deterministic chaos.

The irregularity of both oscillators decreased with increasing Q_n. Their oscillations looked periodic in the region $140 < Q_n < 270$ mV, where both oscillators were entrained to the higher frequency external force (f_{s1}). Figure 3(b) shows the power spectral density obtained at $Q_n = 200$ mV. Few sharp peaks and much low background noise, compared to those in the chaotic state, are observed. The periodic state therefore is induced by noise. Increasing Q_n further, for $Q_n > 270$ mV, both circuits oscillated irregularly and again desynchronized with the external forces. The corresponding Poincaré cross sections are shown in Fig. 3(c) for the chaotic case in the absence of noise and (d) for the periodic case in the presence of noise. In Fig. 3(c), a folding structure can be observed in the Poincaré profile (e.g., in a region C) which indicates "chaos." In addition, the trajectories irregularly traveled among three regions A, B, and C. In Fig. 3(d), on the other hand, the trajectories periodically jumped between A and B and no folding structure was observed.

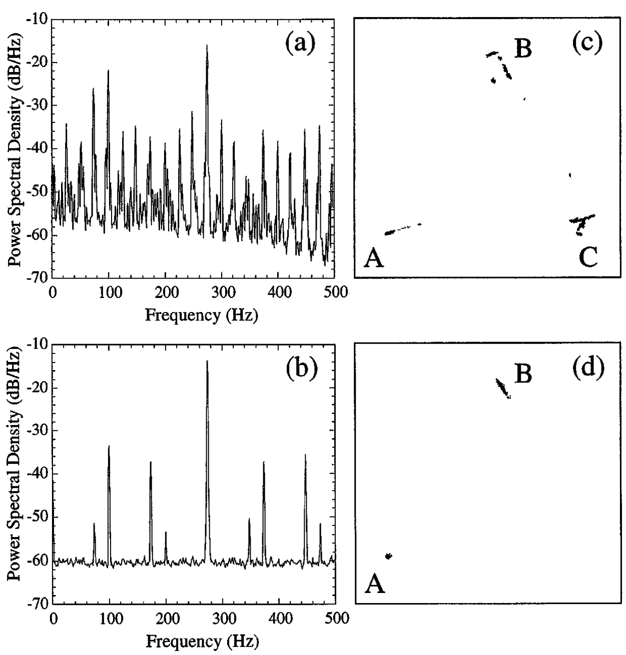

FIG. 3. Power spectral densities [(a) and (b)] and Poincaré cross sections [(c) and (d)] of OS_1. (a) and (c) were obtained in the absence of noise. (b) and (d) were obtained in the presence of noise ($Q_n = 200$ mV). $f_{s1} = 273.973$ Hz, $f_{s2} = 100.000$ Hz, and $g_c = 0.009\,75\ \Omega^{-1}$. The largest peak in the spectrum is at the frequency of the driving force (f_{s1}).

In order to define the irregular motions more quantitatively, λ of the output signal was calculated for various intensities of noise as shown in Fig. 4. The largest Lyapunov exponent $\lambda > 0$ indicating deterministic chaos decreases gradually as the noise intensity increases and at $Q_n = 200$ mV $(= Q_c)$ λ becomes zero which suggests recovery of an ordered state. However, λ is positive value again further increasing the noise intensity beyond Q_c. Thus the result clearly shows the transition from the chaotic state to a nonchaotic one induced by noise. Simultaneously Fig. 4 shows SNR for various noise intensities with λ. SNR of output signals here was obtained using a conventional way such as the ratio of a peak height to intensity of the background noise at the same frequency as the externally applied periodic one (f_{s1}).

SNR becomes better with an increase in the noise intensity, and has a maximum value at $Q_n = 200$ mV, which is just a transition point Q_c in λ. Further increasing Q_n, however, SNR starts to decrease beyond Q_c. This noise dependence of SNR is exactly similar to one of SR phenomena although it shows vice versa. Therefore the resemblance clearly supports the occurrence of SR phenomenon in the present system, and Q_c is the resonance point. Therefore it may be said that NIO observed in the chaos is a kind of SR phenomena.

According to arguments done for the original NIO [21], chaotic behavior depends on details of unstable fixed points which lead to the mixing properties in phase space; that is, it depends on structures of chaotic attractors. If the neighborhood of an unstable fixed point is narrow enough,

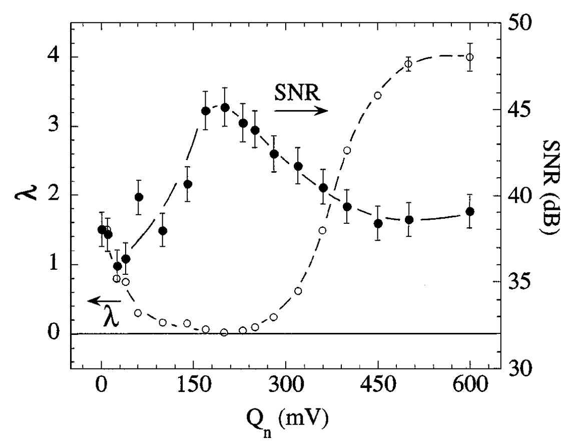

FIG. 4. The largest Lyapunov exponent λ (open circles) and SNR (closed circles). The conditions are the same in Fig. 3. The scatter for λ is smaller than the size of an open circle except for two cases at large Q_n.

the noise may cause the corresponding trajectory jump over it. Then the possibility for the trajectory to approach it becomes small, and the mixing properties therefore may decrease. This results in smaller and even zero λ.

Further increasing noise intensity beyond Q_c, the structure of an unstable fixed point may be modified by large fluctuations of the trajectories due to too strong noise. Therefore the state may become again chaotic with positive and large λ. In the absence of noise, however, the trajectories can always approach an unstable fixed point, and strong mixing takes place. In contrast, if it is an unstable fixed point, which can influence a much larger neighborhood, the noise has less influence on the trajectory, and the probability to skip the neighborhood is smaller. In this case, the trajectory frequently passes near the neighborhood of an unstable fixed point independently of the application of noise.

Consequently, whether NIO may take place or not will largely depend on the types of unstable fixed points, i.e., types of chaotic attractors. Namely, if the chaotic trajectories originally consist of strong localization of positive λ_l (the local Lyapunov exponent), i.e., the nonuniform structure of strange attractors, NIO may happen. But if they consist of global and uniform distribution of positive λ_l, i.e., rather uniform strange attractors, NIO would not happen. The same discussion is possible for SR in chaotic systems.

Based on the preceding discussion, it is now expected that the coupled circuits in the present study form nonuniform attractors in the chaotic state. Then the trajectory related to weak periodic signals externally applied could be more pronounced in the presence of noise. This may lower the Lyapunov exponent. Unfortunately, we cannot directly observe the localization of the trajectory, because embedding to a one-dimensional map has not been done yet.

The dynamical behavior of two coupled chaotic oscillators was experimentally studied under superposition of both the noises and the periodic external forces. We quantitatively determined the chaotic and ordered states by calculating λ and defined the NIO phenomena in a more sophisticated way. We clarified that NIO in chaotic systems was essentially similar to SR. In the present circuits the maximum SNR value was obtained at the smallest value of λ ($\lambda = 0$). Therefore, good agreement was observed between the noise intensity dependencies of SNR and λ. Accordingly, based on the argument regarding NIO, SR in chaotic systems would strongly depend on the structures of chaotic attractors. That is, it may happen that in some chaotic systems SR can be observed but not in others. Finally, we make a remark regarding the future. The application of external noise plays an important role in extracting hidden characteristics in nonuniform chaos, and therefore useful applications can be considered, for example, the diagnosis of chaos by superimposing noise.

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