Anomalous Effective Lagrangian and θ Dependence in QCD at Finite N_c

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We generalize the large N_c Di Vecchia–Veneziano-Witten (VVW) effective chiral Lagrangian to the case of finite N_c by constructing the anomalous effective Lagrangian for QCD. The latter is similar to its supersymmetric counterpart and has a holomorphic structure. The VVW construction is then recovered, along with $1/N_c$ corrections, after integrating out the heavy "glueball" fields. A new mass formula for the η' meson in terms of QCD condensates is obtained. The picture of θ dependence in QCD for finite N_c is more complicated than that predicted by the large N_c approach. [S0031-9007(98)07599-1]

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There exist two different definitions of an effective Lagrangian in quantum field theory. One of them is the Wilsonian effective Lagrangian describing the low energy dynamics of the lightest particles in the theory. Another type of the effective Lagrangian (action) is defined as the Legendre transform of the generating functional for connected Green functions. This object is relevant for addressing the vacuum properties of the theory in terms of vacuum expectation values (VEV's) of composite operators, as they should minimize the effective action. Such an approach is suitable for the study of the dependence of the QCD vacuum on external parameters, such as the light quark masses or the vacuum angle θ. The lowest dimensional condensates $\langle\bar{\Psi}\Psi\rangle$, $\langle G^2\rangle$, $\langle G\tilde{G}\rangle$, which are the most essential for the QCD vacuum structure, are related to the anomalously and explicitly broken conformal and chiral symmetries of QCD. Thus, one can study the vacuum of QCD with an effective Lagrangian realizing at the tree level anomalous conformal and chiral Ward identities of the theory.

The utility of such an approach to gauge theories has been recognized long ago for supersymmetric models, where anomalous effective Lagrangians have been found for both the pure gauge case [1] and supersymmetric QCD [2]. A nonsupersymmetric analog of the Veneziano-Yankielowicz (VY) effective potential [1] was suggested only recently [3]. The purpose of this Letter is to extend the construction [3] to the case of full QCD with N_f light flavors and N_c colors, i.e., to find an effective Lagrangian (more precisely, its potential part) realizing the anomalous conformal and chiral symmetries of QCD.

The interest in such an effective Lagrangian is twofold. First, it provides a generalization of the large N_c Di Vecchia–Veneziano-Witten (VVW) effective chiral Lagrangian (ECL) [4] (see also [5]) for the case of arbitrary N_c after integrating out the massive "glueball" fields. This helps to better understand the origin of the η' mass [the famous U(1) problem]. In particular, it yields a new mass formula for the η' for finite N_c in terms of quark and gluon condensates in QCD [see Eq. (13) below]. Second, such an effective Lagrangian allows one to address the problem of θ dependence in QCD. In contrast to the approach of Ref. [4] which deals from the very beginning with the light chiral degrees of freedom and explicitly incorporates the $U_A(1)$ anomaly without restriction of the topological charge to integer values, in our method both $U_A(1)$ anomaly and topological charge quantization are included in the effective Lagrangian framework. After the glueball fields are integrated out, the topological charge quantization still shows up in the limit $V \to \infty$ via the presence of certain cusps in the effective potential, which are not present in the large N_c ECL of Ref. [4]. Analogous "glued" effective potentials containing cusp singularities arise in supersymmetric $N = 1$ theories when quantization of the topological charge is imposed [6,7]. As will be discussed below, these modifications are not essential for the local properties of the effective chiral potential in the vicinity of the global minimum. In this case, the results of Ref. [4] are reproduced along with calculable $1/N_c$ corrections. On the other hand, for large values of θ and/or ϕ_i our results deviate from those of [4].

We start with recalling the construction [3] of the anomalous effective potential for pure Yang-Mills (YM) theory (gluodynamics). It is constructed as the Legendre transform of the generating functional for zero momentum correlation functions of the marginal operators $G_{\mu\nu}\tilde{G}_{\mu\nu}$ and $G_{\mu\nu}G_{\mu\nu}$ which are related to the gluon condensate due to conformal anomaly [8,9]. The effective potential is a function of effective zero momentum fields $h, \bar{h}$ which describe the VEV's of the composite complex fields $H, \bar{H}$,

$$\int dx\, h = \left\langle \int dx\, H \right\rangle, \qquad \int dx\, \bar{h} = \left\langle \int dx\, \bar{H} \right\rangle, \tag{1}$$

where

$$H(\bar{H}) = \frac{b_{\rm YM}\alpha_s}{16\pi}\left(-G^2 \pm i\,\frac{2}{b_{\rm YM}\xi_{\rm YM}}\, G\tilde{G}\right), \tag{2}$$

Here $b_{\rm YM} = (11/3)N_c$ is the first coefficient of the Gell-Mann–Low β function for YM theory, and $\xi_{\rm YM}$ is a generally unknown rational number, $\xi \equiv q/(2p)$ (here p and q are relatively prime integers), fixing the topological susceptibility in terms of the gluon condensate [3]. On general grounds, it follows that $p = O(N_c)$, $q = O(N_c^0)$. [For some plausible line of reasoning leading to the particular value $\xi = 4/(3b_{\rm YM})$, see [10].] The advantage of using the combinations (2) is in the holomorphic structure of zero momentum correlation functions of $H, \bar{H}$ fields [3,10,11]. As a result, the effective potential also has the holomorphic structure. The final answer for the improved effective potential (IEP) $W(h, \bar{h})$ [here "improved" refers to the necessity of summation over the integers n, k in Eq. (3); see below] reads [3]

$$e^{-iVW(h,\bar{h})} = \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{q-1} \exp\left\{ -\frac{iV}{4} \left(h \,{\rm Log}\, \frac{h}{C_{\rm YM}} + \bar{h} \,{\rm Log}\, \frac{\bar{h}}{\bar{C}_{\rm YM}} \right) + i\pi V \left(k + \frac{q}{p} \frac{\theta + 2\pi n}{2\pi} \right) \frac{h - \bar{h}}{2i} \right\}, \quad (3)$$

where the constants $C_{\rm YM}, \bar{C}_{\rm YM}$ can be taken real and expressed in terms of the vacuum energy in YM theory at $\theta = 0$, $C_{\rm YM} = \bar{C}_{\rm YM} = -2eE_v^{\rm (YM)}(0) = -2e\langle -b_{\rm YM}\alpha_s/(32\pi)G^2\rangle$, and V is the 4-volume. The symbol Log in Eq. (3) stands for the principal branch of the logarithm.

The double sum over the integers n, k in Eq. (3) appears as a resolution of an ambiguity of the effective potential as defined from the anomalous Ward identities WI's. As was discussed in [3], this ambiguity is due to the fact that any particular branch of the multivalued function $h \log(h/c)^{p/q}$, corresponding to some fixed values of n, k, satisfies the anomalous WI's. However, without the summation over the integers n, k in Eq. (3), the effective potential would be multivalued and unbounded from below. An analogous problem arises with the original VY effective Lagrangian. It was cured by Kovner and Shifman in [6] by a similar prescription of summation over all branches of the multivalued VY superpotential. Moreover, the whole structure of Eq. (3) is rather similar to that of the (amended) VY effective potential. Namely, it contains both the "dynamical" and "topological" parts (the first and the second terms in the exponent, respectively). The dynamical part of the effective potential (3) is similar to the VY [1] potential $\sim S \log(S/\Lambda)^{N_c}$ (here S is an anomaly superfield), while the topological part is akin to the improvement [6] of the VY effective potential. Similarly to the supersymmetric case, the infinite sum over n reflects the summation over all integer topological charges in the original YM theory. Thus, the topological charge quantization appears naturally in the effective Lagrangian framework and leads to the single valuedness and boundness from below of the IEP (3).

We now proceed to the generalization of the result (3) to the case of full QCD with N_f light flavors and N_c colors. In the effective Lagrangian approach, the light matter fields are described by the unitary matrix U_{ij} corresponding to the γ_5 phases of the chiral condensate $\langle \bar{\Psi}_L^i \Psi_R^j \rangle = -|\langle \bar{\Psi}_L \Psi_R \rangle| U_{ij}$ with

$$U = \exp\left[i\sqrt{2}\, \frac{\pi^a \lambda^a}{f_\pi} + i\, \frac{2}{\sqrt{N_f}} \frac{\eta'}{f_{\eta'}} \right], \qquad UU^\dagger = 1\,, \quad (4)$$

where λ^a are the Gell-Mann matrices of SU(N_f), π^a is the pseudoscalar octet, and $f_\pi = 133$ MeV. As is well known [4], the effective potential for the U field (apart from the mass term) is uniquely fixed by the chiral anomaly, and amounts to the substitution

$$\theta \to \theta - i\, {\rm Tr}\, \log U \quad (5)$$

in the topological density term in the QCD Lagrangian. Note that for spatially independent vacuum fields U, Eq. (5) results in the shift of θ by a constant. This fact will be used below. Furthermore, in the sense of anomalous conformal Ward identities [8], QCD reduces to pure YM theory when the quarks are "turned off" with the simultaneous substitution $\langle G^2 \rangle_{\rm QCD} \to \langle G^2 \rangle_{\rm YM}$ and $b \equiv b_{\rm QCD} \to b_{\rm YM}$. Analogously, an effective Lagrangian for QCD should transform to that of pure YM theory when the chiral fields U are "frozen." Its form is thus suggested by these arguments and Eqs. (3) and (5),

$$e^{-iVW(h,U)} = \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{q-1} \exp\left\{ -\frac{iV}{4} \left(h \,{\rm Log}\, \frac{h}{2eE} + \bar{h} \,{\rm Log}\, \frac{\bar{h}}{2eE} \right) + i\pi V \left(k + \frac{q}{p} \frac{\theta - i \log {\rm Det}\, U + 2\pi n}{2\pi} \right) \frac{h - \bar{h}}{2i} + \frac{i}{2} V\, {\rm Tr}(MU + {\rm H.c.}) \right\}, \quad (6)$$

where $M = {\rm diag}(m_i |\langle \bar{\Psi}^i \Psi^i \rangle|)$, and the complex fields $h, \bar{h}$ are defined as in Eq. (2) with the substitution $b_{\rm YM} \to b = (11/3)N_c - (2/3)N_f$. We note that all dimensional parameters in Eq. (6) are fixed: $\langle \bar{\Psi}\Psi \rangle \simeq -(240\ {\rm MeV})^3$, $E = \langle b\alpha_s/(32\pi)G^2 \rangle \simeq 0.003\ {\rm GeV}^4$ for $N_c = N_f = 3$ (see below), while the only unknown input is the integers p, q which are in general different from those standing in Eq. (3). They are related to a discrete symmetry surviving the anomaly and can be found only by a direct dynamical calculation. In particular, in supersymmetric (SUSY) theories there exists a special technique [12] relating the strong coupling and weak coupling regimes, which reveals that $p = N_c$, $q = 1$ for SU(N), $p = N_c - 2$, $q = 1$ for SO(N), etc. We are not in a position to calculate these numbers in QCD [see, however, [10] which suggests $q/p = 2\xi = 8/(3b)$], and thus treat them as free parameters. We note that the dynamical part of the anomalous effective potential (6) can be written as $W_d + W_d^\dagger$ where

$$W_d(h,U) = \frac{1}{4}\,\frac{q}{p}\,h\,\mathrm{Log}\left[\left(\frac{h}{2eE}\right)^{p/q}\frac{\mathrm{Det}\,U}{e^{-i\theta}}\right] - \frac{1}{2}\,\mathrm{Tr}\,MU\,, \tag{7}$$

which is quite similar to the effective potential [2] for SUSY QCD [13].

We now wish to argue that Eq. (6) represents the sought-after anomalous effective Lagrangian realizing broken conformal and chiral symmetries of QCD. The following arguments will be suggested: (1) Equation (6) correctly reproduces the VVW ECL [4] in the large N_c limit; (2) Eq. (6) reproduces the anomalous conformal and chiral Ward identities of QCD; (3) it produces a new mass formula for the η' which appears reasonable phenomenologically; (4) it reproduces the known results for the θ dependence at small θ, but may lead to a different behavior for larger values $\theta > \pi/q$ if $q \neq 1$.

(1) The heavy glueball fields $h, \bar{h}$ can be integrated out in Eq. (6) in the same way as was done in [3]. The result is

$$W_{\mathrm{eff}}(U,U^\dagger) = -\lim_{V\to\infty}\frac{1}{V}\log\left\{\sum_l \exp\left[VE\cos\left(-\frac{q}{p}(\theta - i\log\mathrm{Det}\,U) + \frac{2\pi}{p}\,l\right) + \frac{1}{2}\,V\,\mathrm{Tr}(MU + M^\dagger U^\dagger)\right]\right\},$$
$$l = 0,1,\ldots,p-1\,. \tag{8}$$

For small values of $\theta - i\log\mathrm{Det}\,U < \pi/q$ the term with $l = 0$ dominates the infinite volume limit. We obtain for this case,

$$W_{\mathrm{eff}}^{(l=0)}(U,U^\dagger) = -E\cos\left[-\frac{q}{p}(\theta - i\log\mathrm{Det}\,U)\right] - \frac{1}{2}\,\mathrm{Tr}(MU + M^\dagger U^\dagger)\,. \tag{9}$$

Expanding the cosine (with the expansion parameter $q/p \sim 1/N_c$), we recover exactly the ECL of [4] at lowest order in $1/N_c$ (but only for $\theta - i\log\mathrm{Det}\,U < \pi/q$), together with the "cosmological" term $-E = -\langle b\alpha_s/(32\pi)G^2\rangle$ required by the conformal anomaly,

$$W_{\mathrm{eff}}^{(l=0)}(U,U^\dagger) = -E - \frac{\langle\nu^2\rangle_{\mathrm{YM}}}{2}(\theta - i\log\mathrm{Det}\,U)^2 - \frac{1}{2}\,\mathrm{Tr}(MU + M^\dagger U^\dagger) + \cdots, \tag{10}$$

where we used the fact [3] that, at large N_c, $E(q/p)^2 = -\langle\nu^2\rangle_{\mathrm{YM}}$, where $\langle\nu^2\rangle_{\mathrm{YM}} < 0$ is the topological susceptibility in pure YM theory. Corrections in $1/N_c$ stemming from Eq. (9) constitute a new result. Thus, in the large N_c limit the effective chiral potential (8) coincides with that of [4] in the vicinity of the global minimum. At the same time, terms with $l \neq 0$ in Eq. (8) result in different global properties of the effective chiral potential in comparison with the one of Ref. [4]; see below.

(2) It is easy to check that the anomalous chiral and conformal WI's are reproduced by Eq. (6). The anomalous chiral WI's are automatically satisfied by the substitution (5) for any N_c, in accord with [4]. Furthermore, it can be seen that the anomalous conformal WI's of [8] for zero momentum correlation functions of operator G^2 are also satisfied with the above choice of constant E. As an important example, let us calculate the topological susceptibility in QCD near the chiral limit from Eq. (8). For simplicity, we consider the limit of SU(N_f) isospin symmetry with N_f light quarks, $m_i \ll \Lambda_{\mathrm{QCD}}$. For the vacuum energy for small $\theta < \pi/q$ we obtain [see Eq. (16) below]

$$E_{\mathrm{vac}}(\theta) = -E + m\langle\bar\Psi\Psi\rangle N_f\cos\left(\frac{\theta}{N_f}\right) + O(m_q^2)\,. \tag{11}$$

Differentiating this expression twice in θ, we reproduce the result of [14],

$$\lim_{q\to 0} i\int dx\,e^{iqx}\langle 0|T\left\{\frac{\alpha_s}{8\pi}G\tilde{G}(x)\,\frac{\alpha_s}{8\pi}G\tilde{G}(0)\right\}|0\rangle = \frac{1}{N_f}\,m\langle\bar\Psi\Psi\rangle + O(m_q^2)\,. \tag{12}$$

(3) To find a mass formula for the η' meson, we should calculate the matrix of second derivatives at the minimum of the effective potential for small θ. Neglecting for simplicity a small $\pi^0 - \eta - \eta'$ mixing, we obtain

$$f_{\eta'}^2 m_{\eta'}^2 = \frac{8}{9b}N_f\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle - \frac{4}{N_f}\sum_{u,d,s} m_i\langle\bar\Psi_i\Psi_i\rangle + O(m_q^2)\,. \tag{13}$$

[Here we used the value $q/p = 8/(3b)$ suggested by the method of [10]. The choice $p = N_c$, $q = 1$ would produce a numerically close result.] This mass relation for the η' appears reasonable phenomenologically. Note that, according to Eq. (13), the strange quark contributes (30–40)% of the η' mass. In the formal limit $N_c \to \infty$, $m_q \to 0$ Eq. (13) coincides with the relation obtained in [9]. Equation (13) suggests that the main origin of the large mass of the η' is the conformal anomaly in QCD, in accordance with arguments of [3,9].

(4) The main difference of the effective chiral potential (8) from that of [4] is its nonanalytic structure in the limit $V \to \infty$, with cusp singularities at certain values of $\theta - i\log\mathrm{Det}\,U$. The origin of this nonanalyticity is the same as in the pure YM case—it appears when the topological charge quantization is imposed explicitly at the effective Lagrangian level [3]. Thus, the cusp structure of the effective potential seems to be an unavoidable consequence of the topological charge quantization (which was not explicitly imposed in [4]). This fact was first

noted in the context of SUSY theories [6,7], where similar formulas arise.

For practical applications, it is convenient to describe the nonanalytic effective chiral potential (8) by a set of analytic functions defined on different intervals of the combination $\theta - i \log \mathrm{Det}\, U$. Thus, in the infinite volume limit the effective potential for the fields $U = \mathrm{diag}(\exp i\phi_q)$ is dominated by its lth branch,

$$W_{\mathrm{eff}}^{(l)} = -E \cos\left(-\frac{q}{p}\theta + \frac{q}{p}\sum \phi_i + \frac{2\pi}{p} l\right) - \sum M_i \cos \phi_i, \qquad l = 0, 1, \ldots, p-1, \quad (14)$$

if

$$(2l-1)\frac{\pi}{q} \leq \theta - \sum \phi_i < (2l+1)\frac{\pi}{q}. \quad (15)$$

The minimization equations stemming from Eqs. (14) and (15) can be analyzed numerically or analytically in different limits, as was done in [4]. For the case $q = 1$, we find for the locations and number of vacua and the θ dependence of the vacuum energy the same results as those of [4]. In particular, we find the vacuum doubling at $\theta = \pi$ (Dashen's phenomenon [15]) if $m_u m_d > m_s|m_d - m_u|$. As this condition is not satisfied with the physical values of the quark masses, we conclude that in the case $q = 1$ there is the unique vacuum for all values of θ. The only difference in our results from those of [4] for $q = 1$ is a different global form of the effective chiral potential with cusp singularities at certain values of the fields. No metastable local minima of the effective potential exist for any θ (they appear only for $N_f > 4$), although saddle points are there.

If, on the other hand, $q \neq 1$, we find the results for the θ dependence and global structure of the effective chiral potential which are very different from those of Ref. [4]. In particular, in the case of equal masses $m_i = m \ll \Lambda_{\mathrm{QCD}}$ the lowest energy state is described by

$$\phi_i^{(l=0)} = \frac{\theta}{N_f}, \qquad \theta \leq \frac{\pi}{q}, \qquad E_{\mathrm{vac}}(\theta) = -E - MN_f \cos\left(\frac{\theta}{N_f}\right) + O(m_q^2), \quad (16)$$

$$\phi_i^{(l=1)} = \frac{\theta}{N_f} - \frac{2\pi}{qN_f}, \qquad \frac{\pi}{q} \leq \theta \leq \frac{3\pi}{q}, \qquad E_{\mathrm{vac}}(\theta) = -E - MN_f \cos\left(\frac{\theta}{N_f} - \frac{2\pi}{qN_f}\right) + O(m_q^2), \quad (17)$$

etc. Thus, the solution (16) coincides with the one obtained by VVW [4] at small $\theta < \pi/q$ up to $O(m_q^2)$ terms. However, at larger values of θ the true vacuum switches from (16) to (17) with the cusp singularity at $\theta = \pi/q$. (If $q = 1$, the results of [4] are recovered.) The interesting feature of the case $q \neq 1$ is that the vacuum doubling at the points

$$\theta_k = (2k+1)\frac{\pi}{q}, \qquad k = 0, 1, \ldots, p-1 \quad (18)$$

holds irrespective of the values of the light quark masses. This can be seen from the fact that the equations of motion for any two branches with $l = k$ and $l = k+1$ from the set (14) are related by the shift $\theta \to \theta - 2\pi/q$. Thus, the extreme sensitivity of the theory to the values of the light quark masses in the vicinity of the critical point in θ is avoided in our scenario if $q \neq 1$, while the location of the critical point is given by $\theta_c = \pi/q$ instead of the "standard" $\theta_c = \pi$. Another interesting feature of the scenario $q \neq 1$ is the appearance of metastable vacua which exist for any value of θ, including $\theta = 0$. For the physical values of the quark masses, we find $q - 1$ additional local minima of the effective chiral potential, which are separated by barriers from the true physical vacuum of lowest energy. This implies the appearance of domain walls [16]. If true (i.e., if $q \neq 1$), this result can be important for a number of physical problems. A discussion of these issues, as well as a more detailed derivation of the results presented in this Letter, will be given elsewhere [16].

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