Finite Temperature Transport at the Superconductor-Insulator Transition in Disordered Systems

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I argue that the dc limit of the universal crossover function for the conductivity at the superconductor-insulator transition in disordered systems is an analytic function of dimensionality of the system d, with a simple pole at $d = 1$. By combining the exact result for the crossover function in $d = 1$ with the recursion relations in $d = 1 + \epsilon$, the leading term in the Laurent series in ϵ is computed for the systems of disordered bosons. The universal dc critical conductivity for the system with Coulomb interactions in $d = 2$ is estimated to be $\sim 0.69(2e)^2/h$, in satisfactory agreement with many experiments. [S0031-9007(98)07444-4]

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Different quasi-two-dimensional (2D) electronic systems, such as thin films [1], Josephson junction arrays [2], or underdoped high-T_c cuprates [3], appear to have a continuous zero-temperature phase transition between the superconducting and the insulating states, as some parameter of the system is varied. The loss of phase coherence in the ground state is believed to be due to Anderson localization of the Cooper pairs [4,5], which through the quantum uncertainty relation competes with phase ordering. In reality, this purely quantum ($T = 0$) phenomenon is unavoidably observed at low but finite temperatures, typically as a crossover from increasing to decreasing dc conductivity with temperature. Particularly interesting is the behavior of conductivity near the critical value of the tuning parameter, which remains finite down to the lowest temperatures, and usually very close to $(2e)^2/h$, the quantum unit of conductance for electron pairs. This near-universality of the *metallic* transport right at the superconductor-insulator (SI) transitions in 2D has been a subject of numerous theoretical and experimental investigations within the last decade. Scaling arguments [6] imply that at $T = 0$ and right at the quantum critical point the dc conductivity must indeed be universal, very much like the critical exponents. This insight has received a substantial amount of theoretical support in the form of concrete numerical [7–9] and analytical [7,10,11] evaluations of the $T = 0$ critical conductivity for various universality classes of the SI transitions. The experimental evidence for the expected universality of the charge transport, however, is still somewhat less convincing. In particular, measurements typically yield values of the critical conductivity 2 to 3 times larger than the calculations [1]. Most of the theoretical studies [7–11] of the critical conductivity, however, are in significant conceptual discord with the experiment, as recently emphasized by Damle and Sachdev [12]. They argued that the $\omega = 0$, $T \to 0$ dc conductivity that is typically measured and the $T = 0$, $\omega \to 0$ conductivity that is usually calculated originate in entirely different dissipation mechanisms, and, while both should be universal, they have no reason to be equal. As an illustration, they computed the conductivity in the hydrodynamic regime, $\hbar\omega/k_B T \to 0$, at the simpler superfluid-Mott insulator (SF–MI) transition brought by an external periodic potential, near its upper critical dimension of $d = 3$, and indeed found a different, and larger, value of the critical conductivity.

In real disordered systems the insulating phase is presumably the result of Anderson localization and should be a compressible Bose glass (BG) [5], not the Mott insulator. It has been argued that, in the presence of disorder, the limits $\omega \to 0$ and $T \to 0$ may commute [7], and the imaginary time Monte Carlo calculations indeed found little or no dependence of the critical conductivity at the SF-BG transition on the ratio ω_n/T [9], where ω_n is the Matsubara frequency. Including disorder makes the problem difficult to study analytically, since the SI transition then seems to lack the upper critical dimension, or any other obvious limit in which at the criticality the system would be weakly coupled. Numerical techniques [7–9] on the other hand, determine the conductivity by extrapolating from the imaginary Matsubara frequencies, $\omega_n = n2\pi k_B T$, and thus by its nature are not very sensitive to any structure it might have in the hydrodynamic regime. The understanding of the role of disorder and the calculation of the experimentally measured low temperature dc conductivity at the SI critical point in 2D systems thus presents itself as a fundamental and unsolved problem. The purpose of this Letter is to propose its solution in a form of a controlled expansion around the solvable case of the SF-BG transition in one dimension (1D).

I will consider the system of *disordered* interacting bosons of charge $e_* = 2e$ defined, for example, by the Bose-Hubbard, or Josephson junction array Hamiltonian with a random chemical potential [5,11]. This model

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is known to possess a continuous SF-BG transition at $T = 0$, and should be appropriate for description of the SI transition observed in Josephson junction arrays, or ^{4}He in random media. If the Coulomb interaction between bosons is added, it should also represent the correct universality class for the transition in homogeneous and granular films. Although there is no upper critical dimension for the SF-BG transition in the usual sense [5,13,14], the fact that the superfluid phase in $d = 1$ and at $T = 0$ exhibits only a power-law long-range order implies that $d = 1$ represents the lower critical dimension. Recently, this observation has been used by the author to formulate a controlled expansion of the universal quantities at the SF-BG transition in powers of a small parameter $\epsilon = d - 1$ [14]. In particular, and as argued below, in the limit $T \to 0$ the dc resistivity at the critical point is proportional to a certain power of temperature and to the value of the disorder parameter at the SF-BG critical point, which becomes infinitesimally small ($\sim\epsilon$) as dimensionality of the system is reduced to $d = 1$. This suggests that the universal part of the low temperature dc conductivity at the critical point may be expressed as an analytic function of ϵ, with a simple pole at $\epsilon = 0$. I compute the first term in the Laurent series for the real part of the critical dc conductivity around $\epsilon = 0$:

$$\sigma_c'(\omega = 0, T) = \left(\frac{\hbar c}{k_B T}\right)^{[2-d]/z} \left[\frac{6}{\pi^{5/2}}\frac{x}{\epsilon} + O(1)\right]\frac{e_*^2}{h}, \tag{1}$$

where c is a microscopic constant with units $(\text{length})^z/\text{time}$, $z = \{d, 1\}$ is the dynamical exponent, and $x = \{1, 2\}$ for the short range and for the Coulomb interaction between bosons, respectively. For $d = 2$ this leads to an estimate $\sigma_c \approx 0.69 e_*^2/h$ for the Coulomb universality class, in reasonable agreement with many experiments on thin films [1].

In general, right at the critical point in a d-dimensional system the real part of conductivity for frequencies and temperatures much smaller than some microscopic cutoff energy can be written as

$$\sigma_c'(\omega, T) = \left(\frac{\hbar c}{k_B T}\right)^{[2-d]/z} F_d\left(\frac{\hbar\omega}{k_B T}\right)\frac{e_*^2}{h}, \tag{2}$$

where $F_d(x)$ is a *universal* crossover function. It is evident that, in principle, at the critical point the conductivity in $d = 2$ assumes two different values depending on the order of limits $\omega \to 0$ and $T \to 0$, that correspond to the values of the crossover function either at zero or at infinity, and measure completely incoherent or completely coherent transport, respectively. What is more surprising, although it arises as a natural consequence of the scaling law, is that even in the limit $\hbar\omega/k_B T \to 0$ of the incoherent, collision dominated transport, conductivity at the critical point in $d = 2$ is still given by a *universal* number [12]. To understand the dependence of the conductivity in this limit on dimensionality of the system, consider the effective low-energy action for the disordered superfluid that describes the SF-BG transition in the dirty Bose-Hubbard model in $d = 1$ [11,14,15]:

$$S = \frac{K}{\pi}\sum_{i=1}^{N}\int dx \int_0^\beta d\tau \{c^2[\partial_x\theta_i(x,\tau)]^2 + [\partial_\tau\theta_i(x,\tau)]^2\} - D\sum_{i,j=1}^{N}\int dx\int_0^\beta d\tau\, d\tau' \cos 2[\theta_i(x,\tau) - \theta_j(x,\tau')], \tag{3}$$

where $\beta = \hbar/k_B T$, K is inversely proportional to the superfluid density, c is the velocity of low-energy phononic excitations, index i numerates the replicas introduced to average over disorder, and D is related to the width of the distribution of the random potential. The usual limit $N \to 0$ and a short-distance cutoff Λ^{-1} in Eq. (3) are assumed. This effective action arises as the one-dimensional realization of the density (dual) representation of the dirty Bose-Hubbard model at low energies [11], or as the bosonic representation of the disordered Luttinger liquid [15]. The invariance under a change of cutoff implies that the conductivity in $d = 1$ can be written in the scaling form,

$$\sigma(\omega, T) = \frac{\hbar c}{k_B T} f\left(K(b), c(b), D(b), \frac{\hbar\omega}{k_B T}\right)\frac{e_*^2}{h}, \tag{4}$$

where $K(b)$, $c(b)$, and $D(b)$ are the renormalized couplings at the new cutoff $b\Lambda^{-1}$, and $b = \hbar c\Lambda/k_B T$. The result of the renormalization in the theory (3) for weak disorder is well known [14,15]: under the change of cutoff the combination $\kappa = 1/Kc^2$, that is proportional to the compressibility, stays constant, K always increases, and small D is relevant for $\eta = Kc > 1/3$, and irrelevant otherwise. There exists a separatrix in the $\eta - D$ plane which ends in the SF-BG critical point at $\eta = 1/3$ and $D = 0$. At $\omega = 0$ the scaling function in Eq. (4) at weak disorder should behave as $f \sim 1/D(b)$ [15] and therefore right at the separatrix should slowly (logarithmically, $\sim[\ln(b)]^2$) diverge as $b \to \infty$ in $d = 1$. Apart from the logarithmic correction that derives from disorder being dangerously irrelevant, at the SF-BG criticality in 1D, Eq. (4) is just a special case of the general scaling form (2), for $d = z = 1$. Since in $d = 1 + \epsilon$ disorder at the transition scales towards a finite fixed point value $D(b) \to D^* \sim \epsilon$, when $b \to \infty$ [14], the comparison of the two scaling laws in Eqs. (2) and (4) leads to the identification

$$F_{1+\epsilon}(0) = f(1/3, 1, D^*, 0) = \frac{\text{const}}{\epsilon} + O(1), \tag{5}$$

which expresses the central idea of this work. The leading term in the Laurent series for $F_d(0)$ is completely determined by the scaling function as in 1D and by the infinitesimal value of disorder at the fixed point of the scaling transformation in $d = 1 + \epsilon$.

In the remainder of the paper, the entire function f near the SF-BG critical point in $d = 1$ is obtained, a new field-theoretic version of the recursion relations in $d = 1 + \epsilon$

requisite for the determination of the fixed point value of disorder is derived, and, finally, the residuum at the pole at $\epsilon = 0$ in Eq. (5) for both short-range and Coulomb interactions between bosons is computed.

The standard linear response formalism yields the conductivity in $d = 1$:

$$\sigma(\omega, T) = -i \frac{2\omega}{\pi} \frac{e_*^2}{h} \lim_{N\to 0} \frac{1}{N} \sum_{n,m=1}^{N} G_{nm}^r(\omega). \quad (6)$$

$G_{nm}^r(\omega)$ is the temperature-dependent, retarded, $q = 0$ Green's function defined as

$$G_{nm}^r(\omega) = \int dx \int_0^\beta d\tau\, e^{i\omega_n \tau} \times \langle T_\tau \theta_n(x,\tau)\theta_m(0,0)\rangle_{i\omega_n \to \omega + i\delta}, \quad (7)$$

where T_τ is the standard time-ordering operator. The thermal Green's function in Eq. (7) may be evaluated perturbatively in disorder and then analytically continued to real frequencies. A similar calculation has been performed before by Luther and Peschel [16] for $\eta > 1$, which corresponds to weak coupling in the equivalent 1D fermionic system. The SF-BG transition at weak disorder is at $\eta \approx 1/3$, so I derive here a slightly improved version of their results which can be analytically continued into the transition region. Introduce the thermal self-energy as $G_{ij}^t(\omega_n) = \delta_{ij}/[2(K/\pi)\omega_n^2 + \Sigma^t(\omega_n)]$. To the lowest order in D it may be written as

$$\Sigma^t(\omega_n) = 8D \int_0^\beta d\tau\, (1 - e^{i\omega_n\tau}) \langle T_\tau e^{i2\theta_j(x,\tau)} e^{-i2\theta_j(x,0)}\rangle_0 + O(D^2), \quad (8)$$

where the average is performed over the quadratic part of the action (3). The self-energy is thus *itself* a Green's function, which enables one to perform the analytic continuation to real frequencies by first rotating the integrand in (8) to real time by $\tau \to it$ to find the real time time-ordered propagator, and from it finally to determine the retarded one by using the standard relation between them [17]. Performing the Fourier transform at the resulting expression then gives the retarded self-energy

$$\Sigma^r(\omega) = \frac{16\pi D}{c\Lambda} \left(\frac{\pi k_B T}{\hbar c \Lambda}\right)^{(1/\eta)-1} \sin\!\left(\frac{\pi}{2\eta}\right) e^{C/\eta} \times \int_0^\infty dt\, \frac{1 - e^{i(\hbar\omega/k_B T)t}}{[\sinh(\pi t)]^{1/\eta}}, \quad (9)$$

where $C \approx 0.577$ is the Euler's constant, and I assumed that $\hbar c\Lambda/k_B T \gg 1$, i.e., the continuum (or equivalently, the low-temperature) limit. The appearance of a particular numerical constant is a consequence of the assumption that the dispersion in (3) is $\omega = ck$ for all momenta $0 < k < \Lambda$, and is a nonuniversal, short-distance feature. This, nevertheless, does not compromise the universality of the conductivity at the transition, since any nonuniversal constant such as C may at the end be absorbed into the definition of the running, dimensionless disorder coupling, as will be done shortly. The remaining integral in (9) is convergent only for $\eta > 1$, but, once evaluated there exactly, may be defined in the transition region $\eta \approx 1/3$ via analytic continuation. Performing the integral, in the vicinity of $\eta = 1/3$ one obtains the conductivity in $d = 1$ to be

$$\sigma(\omega, T) = \frac{i\omega c}{\eta(T)\omega^2 + i2(k_B T/\hbar)^2 W(T) g(\hbar\omega/k_B T)} \frac{e_*^2}{h}, \quad (10)$$

where $W(T) = (\pi^4 D/c^2\Lambda^3 e^{3C})(k_B T/\hbar c\Lambda)^{(1/\eta)-3}$ is the dimensionless disorder variable, $\eta(T) = \eta + W(T)\pi^{-3/2}\tan(\pi/2\eta)$, and $g(x) = [1 + (x/\pi)^2] \times \tanh(x/2)$. Note that the result indeed may be cast into the scaling form as claimed in Eq. (4). The self-energy acquired an imaginary part, which in the limit $\hbar\omega/k_B T \to 0$ becomes proportional to temperature and to the temperature-dependent disorder variable $W(T)$. The significance of the point $\eta = 1/3$ now becomes apparent: for $\eta < 1/3$ the disorder-variable $W(T)$ scales towards zero with decrease in temperature, and the lowest-order result in Eq. (10) becomes asymptotically exact. At a finite frequency and at $T = 0$ the real part of conductivity in $d = 1$ then becomes $\sigma'(\omega) = [\pi c/\eta(0)]\delta(\omega)$, and the system is an ideal conductor. If $\eta > 1/3$ the perturbation theory breaks down, which indicates the entrance into the insulating phase. Notice that as $\eta \to 1/3^-$ the coefficient in front of $W(T)$ in the expansion for $\eta(T)$ becomes divergent, as it has a simple pole at $\eta = 1/3$. This is reminiscent of the dimensional regularization frequently employed in the studies of thermal critical phenomena, and suggests that the theory (3) becomes just renormalizable at $\eta = 1/3$.

In the continuum limit, the effect of change of temperature on the low-frequency conductivity in $d = 1$ may be expressed entirely through the effective values of the coupling constants $\eta(T)$, $W(T)$, and, if we had retained the momentum dependence of the propagator, the compressibility $\kappa(T)$. Taking into account the finite canonical dimensions of the coupling constants η and κ away from $d = 1$ [14], close to $\eta = 1/3$ the effective couplings satisfy the differential equations:

$$\dot{\eta}(T) = z^{-1}(d-1)\eta(T) - \frac{2}{\pi^{5/2}} W(T) + O(W^2(T)), \quad (11)$$

$$\dot{W}(T) = (\eta^{-1} - 3)W(T) + O(W^2(T)), \quad (12)$$

$$\dot{\kappa}(T) = z^{-1}(z-d)\kappa(T), \quad (13)$$

where $\dot{x} = dx/d\ln(k_B T/\hbar c\Lambda)$. The d-dependent terms in Eqs. (11) and (13) follow from the scaling of the superfluid density $\rho_{sf} \sim K^{-1} \sim \xi^{2-z-d}$ and the compressibility $\kappa \sim 1/Kc^2 \sim \xi^{z-d}$ near the critical point (where $T \sim \xi^{-z}$, and ξ is the diverging correlation length). These scaling relations can be directly read from the low-energy theory for the superfluid [5], as discussed at length elsewhere [14]. Note that in Eq. (13), unlike in Eq. (11), there are no terms dependent on $W(T)$. This is a consequence of the exact symmetry of the interaction term in the action (2) under $\theta_i(x,\tau) \rightarrow \theta_i(x,\tau) + h(x)$, for arbitrary function $h(x)$, and fixes the value of dynamical exponent to $z = d$ for the system with short-range interactions [14]. Linearization of the flow close to the fixed point of Eqs. (11) and (12) gives the correlation length exponent $\nu = (1/\sqrt{3\epsilon}) + O(1)$, in agreement with the result of the momentum-shell renormalization group [14]. The fixed point is located at $W^*(T) = \pi^{5/2}\epsilon/6 + O(\epsilon^2)$ and $\eta^*(T) = 1/3 + O(\epsilon)$. Using Eqs. (10) and (5) one obtains the main result announced in Eq. (1), for the short-range interactions between bosons.

To make a comparison with the experiments on thin films [1] or on high-T_c cuprates [3] one must include the long-range Coulomb repulsion between the electron pairs. A way to do this was proposed previously by the author in Ref. [14], where the long-range interactions between the bosons was defined as $V(\vec{r}) = e^2 \int d^d\vec{q} \exp(i\vec{q} \cdot \vec{r})/q^{d-1}$, so that it coincides with the Coulomb interaction for $d > 1$, and with the short-range interaction precisely at $d = 1$. The calculation of the conductivity in $d = 1$ then remains the same, but the recursion relations (11)–(13) need to be modified in two ways [14]. First, instead of the equation for compressibility, one now has the equation for the temperature-dependent charge $e^2(T)$, which has the exactly same form as Eq. (13), except for $(z - d) \rightarrow (z - 1)$. Consequently, since now it is $e^2 \sim 1/Kc^2$, the coupling $\eta = Kc \sim T^{(1-d)/2z}$, and the first term in Eq. (11) is twice smaller. It then follows that with the Coulomb interactions present $z = 1$, $\nu = \sqrt{2/3\epsilon} + O(1)$, and the fixed point value $W^*(T) = \pi^{5/2}\epsilon/12 + O(\epsilon^2)$. This yields the second result quoted in Eq. (1), for the Coulomb universality class. To the lowest order, the critical dc conductivity is larger if the Coulomb interaction is present. This may have been intuitively expected: a longer-range interaction suppresses the phase order more efficiently, so it takes less disorder to finally turn the system into an insulator.

Although there is no very good agreement on the value of the critical conductivity between different experiments, most of the measurements [1] on thin films are very close to $\sigma_c \approx 1(2e)^2/h$. Recent measurements [18] performed on amorphous bismuth films at temperatures below 0.5 K yield $\sigma_c \simeq 0.86(2e)^2/h$, in encouraging agreement with my lowest-order estimate for the Coulomb universality class. It is also interesting to note that for both universality classes the critical conductivity in the hydrodynamic regime obtained here turns out to be larger than the one in the coherent, $T = 0$ limit [9,11], similarly as in the SF-MI transition in a commensurate periodic potential [12]. By continuity, Eq. (10) implies that the crossover function $F_d(x)$ is a continuously decreasing function of its argument for $d = 1 + \epsilon$, with a maximum $\sim 1/\epsilon$ at $x = 0$, and vanishing as $\sim \epsilon/x^{(1-\epsilon)/z}$ for large x. Although not ruled out, it does seem unlikely that in $d = 2$ this dependence on $x = \hbar\omega/k_BT$ completely disappears. In fact, the difference in the estimated critical conductivities in completely coherent and incoherent regimes suggests that the situation in $d = 2$ is likely to be qualitatively similar to the one in $d = 1 + \epsilon$: the real part of conductivity should continuously decrease as a function of $\hbar\omega/k_BT$, interpolating between the two finite, $\omega = 0$ and $T = 0$, limits.

Finally, the field-theoretic formulation of the renormalization group transformation derived here has the advantage over the usual momentum-shell calculation [14,15] in that it facilitates a systematic higher-order calculation. The observation that η in $d = 1$ plays a role similar to dimensionality in the classical critical phenomena suggests a procedure analogous to the standard dimensional regularization for the d-independent part of the recursion relations. Adding the effect of dimensionality when $d > 1$ as described in Ref. [14] and as done in Eqs. (11)–(13) would then yield the higher-order corrections for the exponent ν and the critical dc conductivity. It would be very interesting to compare the results of such an analytical calculation with the experiments and the numerical simulations, as it could lead to a more definite understanding of the SF-BG quantum critical behavior.

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