

Effect of Nonlinear Gradient Terms on Breathing Localized Solutions in the Quintic Complex Ginzburg-Landau Equation

Robert J. Deissler^{1,2} and Helmut R. Brand^{1,3}

¹Center for Nonlinear Studies, MS-B 258, Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545

²Innovative Technologies, 4540 W. 213 Street, Fairview Park, Ohio 44126

³Theoretische Physik III, Universität Bayreuth, D95440 Bayreuth, Germany

(Received 8 January 1998)

We study the effect of nonlinear gradient terms on breathing localized solutions in the complex Ginzburg-Landau equation. It is found that even small nonlinear gradient terms—which appear at the same order as the quintic term—can cause dramatic changes in the behavior of the solution, such as causing opposite sides of an otherwise monophasic symmetrically breathing solution to breathe at different frequencies, thus causing the solution to breathe periodically or chaotically on only one side or the solution to rapidly spread. [S0031-9007(98)07488-2]

PACS numbers: 47.20.Ky, 03.40.Gc, 03.40.Kf, 05.70.Ln

For over thirty years now it has been known that stable localized solutions can exist for certain nonlinear partial differential equations. The best-known example of such solutions is the soliton [1,2], a localized solution which occurs in purely dispersive systems such as the nonlinear Schrödinger equation. More recently stable localized solutions have been found to occur in quintic complex Ginzburg-Landau (CGL) equations [3–7]—generic equations with both dissipation *and* dispersion and which describe systems near a subcritical bifurcation to traveling waves. These dissipative-dispersive localized (DDL) solutions can be considered to be the analog of the solitons that occur in purely dispersive systems. Although these DDL solutions share some properties with solitons, such as a fixed shape for the modulus and interaction behavior in which shape and size are preserved during collisions [4,5], there are fundamental differences. For example, these DDL solutions also exhibit mutual annihilation during collisions [4,5], a property which does not occur for solitons. Also, in contrast to solitons which require no energy input for their existence, the DDL solutions depend on a constant influx of energy in order to overcome the dissipation. Stable DDL solutions have also been studied in a two-dimensional (2D) quintic CGL equation [3,8], in a 2D equation for systems with broken rotational symmetry [9], and in equations describing systems in nonlinear optics—a dye laser with saturable absorber [10] and a system exhibiting optical bistability [11]. Experimentally, stable DDL solutions have been found in binary fluid convection [12,13] and in a dye laser with saturable absorber [14].

Until recently the behavior of localized solutions of prototype equations has been limited to solutions with fixed modulus such as the solitons and DDL solutions discussed above, or to solutions which oscillate periodically about zero (for real equations) such as the “breathers” of the sine Gordon equation. Therefore, an interesting discovery was that of stable localized solutions for which the modulus breathes periodically, quasiperiodically, or even

chaotically [15]. By stable is meant that the solution lies on an attractor. These breathing DDL solutions, which were found for the quintic CGL equation, bore no relation to the “breathers” of the sine-Gordon equation. Also they are very different from the slowly spreading chaotic localized solutions of the quintic CGL equation [16,17]. The breathing DDL solutions exhibit interesting interaction behavior such as dependence on initial conditions for the outcome of collisions and even *sensitive* dependence on initial conditions for the outcome of collisions involving *chaotic* breathing DDL solutions [18].

For the breathing DDL solutions of the quintic CGL equation, nonlinear gradient terms have thus far been neglected for simplicity. However, for an actual physical system nonlinear gradient terms will always be present since they occur at the same order as the quintic term [19]. Therefore, an important question is whether there are any qualitative changes in the behavior as a result of the nonlinear gradient terms.

In this Letter we study the effect of nonlinear gradient terms on the breathing DDL solutions of the quintic CGL equation. We find that even small nonlinear gradient terms can dramatically alter the behavior of solutions, such as causing opposite sides of an otherwise monophasic symmetrically breathing solution to breathe at different frequencies, causing the solution to breathe periodically or chaotically on only one side, or causing the solution to rapidly spread. We find that it is also possible for nonlinear gradient terms to cause an otherwise fixed-shape solution to breathe periodically or even chaotically.

The quintic CGL equation with nonlinear gradient terms reads

$$A_t + vA_x = \chi A + \gamma A_{xx} - \beta |A|^2 A - \delta |A|^4 A - \lambda |A|^2 A_x - \mu A^2 A_x^*, \quad (1)$$

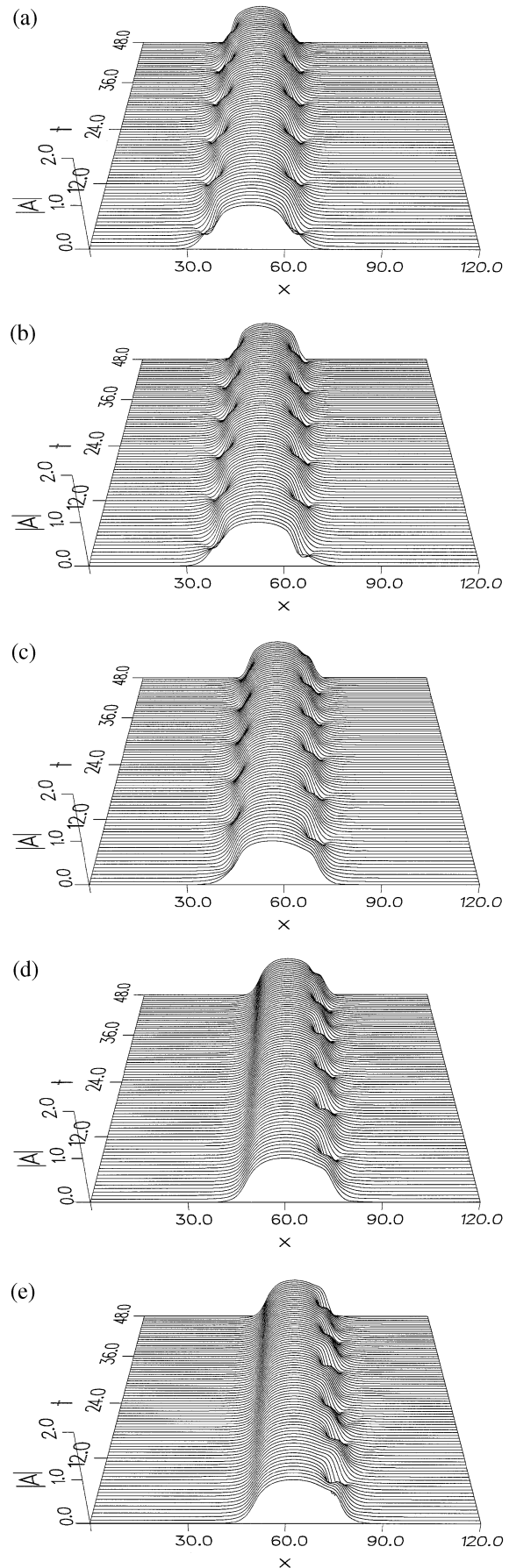
where A is a slowly varying complex amplitude and the coefficients (except for the group velocity v) are in general complex, i.e., of the form $z = z_r + iz_i$. Since we take

periodic boundary conditions, the convective term vA_x may be transformed away by going into a moving frame of reference. Also, the parameter χ , which is proportional to the distance from criticality, can be taken as real, since the imaginary part can be transformed away by a simple transformation. We take $\chi < 0$ and $\beta_r < 0$ so that the system is subcritical and take $\delta_r > 0$ to guarantee saturation. The last two terms are the nonlinear gradient terms. For fixed-shape solutions, these terms cause the solution to become asymmetric and to move at a velocity other than the group velocity [5]. We note that Eq. (1) is the most general equation to the lowest consistent order in an envelope expansion near onset (as discussed first in the appendix of Ref. [19(b)]) and is thus generic in nature. We also stress that there are no other terms that occur at this order. For slightly negative values of β_r , the scaling of the amplitude is determined by the quintic saturation term. The scaling for the spatial derivatives is unchanged, thus ruling out higher order derivative terms linear in the amplitude A .

Figure 1 shows space-time plots of the modulus of A for various values of the nonlinear gradient terms, keeping the other parameter values fixed. The numerical method used was a time-splitting method [20]. The linear terms are integrated exactly in time using Fourier transforms, and the nonlinear terms are integrated using second-order Runge-Kutta, fourth-order spatial differencing being used for the nonlinear gradient terms. The time step is 0.01 and the number of Fourier modes and grid points is 1024. The state was prepared by first starting with a Gaussian centered about $x = 50$ and integrating in time (without the nonlinear gradient terms present) until the solution had settled onto its attractor. The nonlinear gradient terms were then applied and the code was run for an additional 10^5 iterations, giving the system plenty of time to settle onto its attractor. The end of this time period corresponds to $t = 0$ in the plots. The code was then run for an additional 4800 iterations to provide the data for the plots.

For comparison purposes, Fig. 1(a) shows the solution with coefficients of the nonlinear gradient terms equal to zero, i.e., $\lambda = \mu = 0 + 0i$. This solution lies on a periodic attractor with period 1. The waves travel symmetrically down the left and right sides of the solution.

FIG. 1. Space-time plots of breathing DDL solutions for various values of the coefficients of the nonlinear gradient terms. For all the plots in this Letter, the parameter values, other than those for the nonlinear gradient terms, are $\chi = -0.1$, $v = 0$, $\gamma = 1.0 - 1.1i$, $\beta = -3.0 - 1.0i$, and $\delta = 2.75 - 1.0i$. (a) $\lambda = \mu = 0 + 0i$ (symmetric monoperoic solution). (b) $\lambda = \mu = 0.01 + 0.01i$ ("beating solution." The right side breathes at a slightly larger frequency than the left). (c) $\lambda = \mu = 0.03 + 0.03i$ [beating solution. The "beating" frequency is larger, causing opposite sides to be more out of phase with one another as compared with Fig. 1(b).]. (d) $\lambda = \mu = 0.05 + 0.05i$ (period-2 solution). (e) $\lambda = \mu = 0.07 + 0.07i$ (chaotic solution).



The solution remains localized because the system is subcritical and the waves that are shed from the sides are of sufficiently small amplitude that they quickly decay. Figure 1(b) shows the solution with $\lambda = \mu = 0.01 + 0.01i$. As can be seen, the solution is no longer symmetric. The reason for this asymmetry is that, as a result of the nonlinear gradient terms, the wave traveling down the right of the solution is traveling slightly faster than the wave traveling down the left of the solution. Therefore, there are now two distinct frequencies—the frequency of the wave traveling down the right side of the solution and the frequency of the wave traveling down the left. The difference between these two frequencies corresponds to the waves on opposite sides of the solution oscillating in and out of phase with one another. We note that the solution is also traveling slowly to the right.

For larger values of λ and μ , the difference between the frequencies of the right and left waves will become larger. This can be seen by comparing Fig. 1(c), where $\lambda = \mu = 0.03 + 0.03i$, with Fig. 1(b), noting that the right and left waves are more out of phase in Fig. 1(c) as compared to Fig. 1(b) and recalling that the code was run for 10^5 iterations with the nonlinear gradient terms present prior to plotting.

In addition to the frequency difference, referring to Fig. 1(c) it is also seen that the wave traveling down the right of the solution is larger in amplitude than that traveling down the left. For larger nonlinear gradient terms, as seen in Fig. 1(d) ($\lambda = \mu = 0.05 + 0.05i$) and Fig. 1(e) ($\lambda = \mu = 0.07 + 0.07i$), waves are seen to travel only down the right side of the solution. Note also that the solutions travel at a larger velocity to the right for larger nonlinear gradient terms, recalling that the solutions were centered about $x = 50$, 10^5 iterations prior to plotting.

To characterize the time evolution of the solutions, we plot the area under the modulus, $S = \int dx |A|$, as a function of time. We note that this quantity is independent of the velocity at which the solution travels, i.e., Galilean invariant. Figure 2 shows a plot of $S(t)$ for $\lambda = \mu = 0.03 + 0.03i$. These are the same parameters as used in Fig. 1(c). The slow “beating” frequency is equal to the difference between the breathing frequencies of the right and left sides of the solution and corresponds to opposite sides of the solution breathing in and out of phase with one another.

Plotting $S(t)$ for the parameter values of Fig. 1(d) ($\lambda = \mu = 0.05 + 0.05i$), we find that this solution is period 2; i.e., it takes two oscillations for one complete period. Figure 3 shows a plot of $S(t)$ for $\lambda = \mu = 0.06 + 0.06i$. It is seen that this solution is period 4. Plotting $S(t)$ for $\lambda = \mu = 0.064 + 0.064i$, we find a period-8 solution. As a function of the magnitude of the coefficients of the nonlinear gradient terms, the system appears to be undergoing a period-doubling route to chaos. In fact, the space-time plot in Fig. 1(e) looks

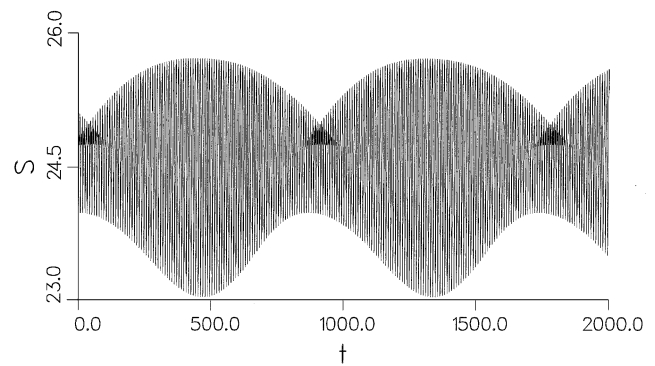


FIG. 2. Time series of the area under the modulus of the breathing DDL solution plotted in Fig. 1(c). Note the beating caused by the left and right sides breathing in and out of phase with one another.

like it could be chaotic. Figure 4(a) shows a plot of $S(t)$ for these parameter values ($\lambda = \mu = 0.07 + 0.07i$). To establish that the solution is chaotic, we plot $\ln(\zeta)$ as a function of time, where $\zeta = (\int dx |\delta A|^2)^{1/2}$, δA being the linear perturbation about A [16]. Figure 4(b) shows a plot of $\ln[\zeta(t)]$. It is seen that nearby solutions separate exponentially on the average and that, therefore, the solution is indeed chaotic. For sufficiently large values of λ and μ , e.g., $\lambda = \mu = 0.08 + 0.08i$, the waves that are shed from the right side of the solution become large enough so that they no longer decay, but instead grow, and the solution no longer remains localized but quickly spreads to the right.

We have seen that small nonlinear gradient terms can dramatically alter a monoperiodic solution, causing opposite sides of the solution to breathe at different frequencies or causing the solution to become period n or chaotic. An interesting question is whether nonlinear gradient terms can have similarly significant effects on fixed-shape solutions. We indeed find this to be the case; a fixed shape solution in the absence of nonlinear gradient terms can become monoperiodic, period n , or even chaotic in the presence of nonlinear gradient terms. For example, taking $\gamma_r = 1.2$ instead of $\gamma_r = 1.0$, we

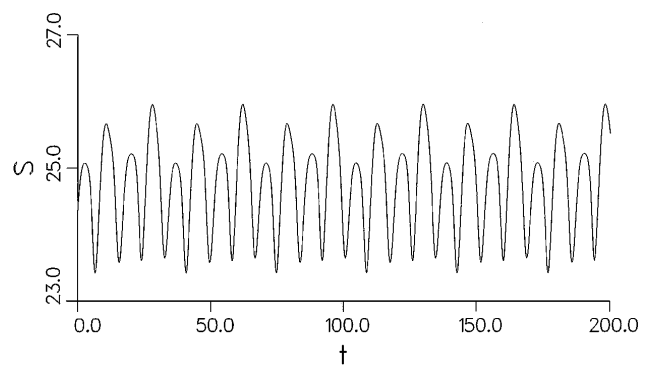


FIG. 3. Time series of the area under the modulus for $\lambda = \mu = 0.06 + 0.06i$. The solution is period 4.

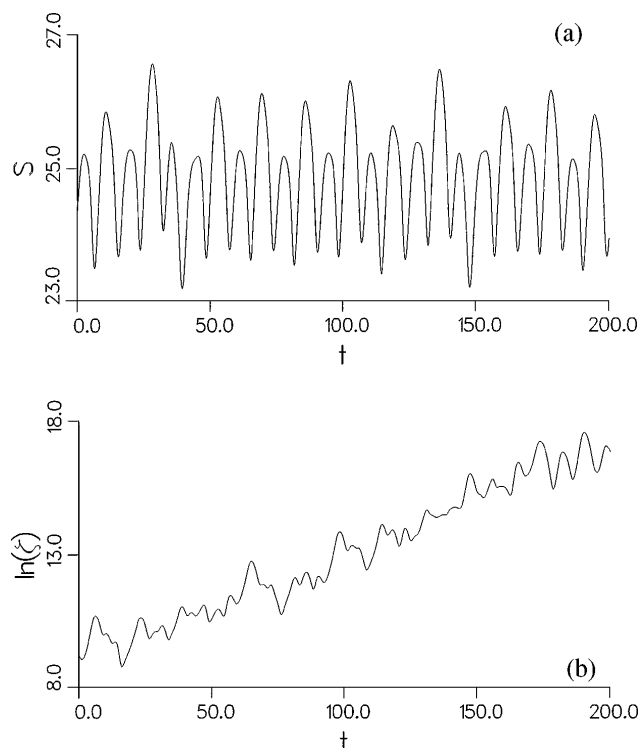


FIG. 4. Time series for the breathing DDL solution plotted in Fig. 1(e). (a) Time series of the area under the modulus. (b) Time series of the logarithm of the separation between nearby states. The solution is clearly seen to be chaotic.

find a fixed shape for $\lambda = \mu = 0 + 0i$, period 1 for $\lambda = \mu = 0.10 + 0.10i$, period 2 for $\lambda = \mu = 0.15 + 0.15i$, period 4 for $\lambda = \mu = 0.16 + 0.16i$, and chaos for $\lambda = \mu = 0.164 + 0.164i$. We note that these breathing solutions breathe only on the right side. Again we find that for sufficiently large nonlinear gradient terms, e.g., $\lambda = \mu = 0.167 + 0.167i$, the solution no longer remains localized but quickly spreads to the right. Thus it emerges that the features of the fixed shape solutions, as well as those of the breathing localized solutions, are changed substantially with increasing magnitude of the nonlinear gradient contributions.

In conclusion, we have studied the effect of nonlinear gradient terms on breathing DDL solutions for the quintic CGL equation. This is important since nonlinear gradient terms appear at the same order as the quintic term. We find that even small nonlinear gradient terms can dramatically alter the behavior of the solution, causing opposite sides of an otherwise monophasic DDL solution to breathe at different frequencies or causing the solution to become period n or chaotic, causing an otherwise symmetrically breathing solution to breathe on only one side, or causing an other-

wise fixed-shape DDL solution to become monophasic, period n , or chaotic. We emphasize that the behavior analyzed here is generic and not restricted to a small parameter regime. It will be interesting to check the predictions experimentally. Candidates include binary fluid convection and electroconvection in nematic liquid crystals.

The work done at the Center for Nonlinear Studies, Los Alamos National Laboratory has been performed under the auspices of the United States Department of Energy. H. R. B. thanks the Deutsche Forschungsgemeinschaft for partial support of this work through the Graduiertenkolleg "Nichtlineare Spektroskopie und Dynamik."

-
- [1] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
 - [2] A. C. Newell, *Solitons in Mathematics and Physics* (Society for Industrial and Applied Mathematics, Philadelphia, PA, 1985).
 - [3] O. Thual and S. Fauve, J. Phys. (Paris) **49**, 1829 (1988).
 - [4] H. R. Brand and R. J. Deissler, Phys. Rev. Lett. **63**, 2801 (1989).
 - [5] R. J. Deissler and H. R. Brand, Phys. Lett. A **146**, 252 (1990).
 - [6] V. Hakim, P. Jakobsen, and Y. Pomeau, Europhys. Lett. **11**, 19 (1990).
 - [7] P. Marcq, H. Chaté, and R. Conte, Physica (Amsterdam) **73D**, 305 (1994), and references therein.
 - [8] R. J. Deissler and H. R. Brand, Phys. Rev. A **44**, R3411 (1991).
 - [9] R. J. Deissler and H. R. Brand, Phys. Rev. E **51**, R852 (1995).
 - [10] H. R. Brand and R. J. Deissler, Physica (Amsterdam) **204A**, 87 (1994).
 - [11] H. R. Brand and R. J. Deissler, Physica (Amsterdam) **216A**, 288 (1995).
 - [12] J. J. Niemela, G. Ahlers, and D. S. Cannell, Phys. Rev. Lett. **64**, 1365 (1990).
 - [13] P. Kolodner, Phys. Rev. Lett. **69**, 2519 (1992); Phys. Rev. A **44**, 6448 (1991); **44**, 6466 (1991).
 - [14] V. B. Taranenko, K. Staliunas, and C. O. Weiss, Phys. Rev. A **56**, 1582 (1997).
 - [15] R. J. Deissler and H. R. Brand, Phys. Rev. Lett. **72**, 478 (1994).
 - [16] R. J. Deissler, J. Stat. Phys. **54**, 1459 (1989).
 - [17] R. J. Deissler, Phys. Lett. A **120**, 334 (1987).
 - [18] R. J. Deissler and H. R. Brand, Phys. Rev. Lett. **74**, 4847 (1995).
 - [19] H. R. Brand, P. S. Lomdahl, and A. C. Newell, (a) Phys. Lett. A **118**, 67 (1986); (b) Physica (Amsterdam) **23D**, 345 (1986).
 - [20] R. J. Deissler, Y. C. Lee, and H. R. Brand, Phys. Rev. A **41**, 2101 (1990).