Clustering and Non-Gaussian Behavior in Granular Matter

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We investigate the properties of a model of granular matter consisting of N Brownian particles on a line, subject to inelastic mutual collisions. This model displays a genuine thermodynamic limit for the mean values of the energy, and the energy dissipation. When the typical relaxation time τ associated with the Brownian process is small compared with the mean collision time τ_c the spatial density is nearly homogeneous and the velocity probability distribution is Gaussian. In the opposite limit $\tau \gg \tau_c$ one has strong spatial clustering, with a fractal distribution of particles, and the velocity probability distribution strongly deviates from the Gaussian one. [S0031-9007(98)07496-1]

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In the past few years granular materials have become an intriguing subject of research [1–7], since they pose novel questions and challenges to the theorists and experimentalists. The constituting elements of such materials are solid particles, whose size may range from a few microns to a few centimeters, and which are subject to nonconservative contact forces such as friction and cohesion.

Their collective behavior is peculiar and different from other forms of matter, such as solids, liquids, or gases, and the ordinary statistical mechanical approach, which successfully deals with large assemblies of microscopic particles, is not adequate.

Generally speaking, granular materials cannot be described as equilibrium systems either from the configurational point of view or from the dynamical point of view. It is known in fact that these systems remain easily trapped in some metastable configurations which can last for long time intervals unless they are shaken or perturbed [2]. On the other hand while in equilibrium statistical mechanics the kinetic energy per particle is proportional to the temperature and the velocities are Gaussianly distributed, in the systems we consider the tails of the distribution deviate from the Maxwell law [8]. This phenomenon is accompanied by a pronounced clustering of the particles [3,4] or *inelastic collapse*.

Several approaches have been proposed for the study of the so-called "granular gases" [7,9]. One crucial difference between ordinary gases and granular media is represented by the intrinsic inelasticity of the interactions among the grains, which makes any theory based on energy conservation, e.g., for ideal gases, not suitable.

In the present work we study a one dimensional mechanical model, in the spirit of the one recently introduced by Kadanoff and co-workers [9], but containing some important differences regarding the energy-exchange process. We consider N identical particles on a circle of length L [10] obeying to the following equations:

$$\frac{dv_i}{dt} = -\frac{v_i}{\tau} + \sqrt{\frac{2T_F}{\tau}}\, f_i(t)\,, \tag{1}$$

$$\frac{dx_i}{dt} = v_i(t)\,, \tag{2}$$

where, $1 \leq i \leq N$, T_F is the temperature of a microscopic medium that we discuss below, τ is the relaxation time, in the absence of collisions, and $f_i(t)$ is a standard white noise with zero average and variance $\langle f_i(t) f_j(t') \rangle = \delta_{ij}\delta(t - t')$. In addition the particles are subject to inelastic collisions according to the rule $v_i' - v_j' = -r(v_i - v_j)$, where r is the restitution coefficient ($r = 1$ for the completely elastic case) [11].

The introduction of the viscous term takes into account important factors, generally disregarded in simplified models, namely the friction among particles and energy transfers among different degrees of freedom, which are relevants in real granular systems. The damping and noisy terms are very natural when interactions between particles and the environment (particle-fluid interactions) start to be important. Another important class of phenomena in which a viscous damping and a noisy term are naturally present is represented by the fluidized beds, where the vibration of the bottom of the box produces a random force on the particles [1,2] and [12].

The model above differs from the one proposed by Kadanoff and co-workers because in the latter the particles, subject to inelastic collisions, are confined to an interval of length L by two asymmetric walls: the first reflecting them elastically and the second supplying energy to the particles according to a Gaussian distribution

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at fixed temperature. As evident from the simulation of Ref. [9] one observes a somehow trivial clustering of the particles next to the elastic wall. We found numerically a more serious shortcoming of such a model consisting in the fact that the average energy and the average energy dissipated per particle, defined respectively as the time average of $E(t) = 1/2 \sum_{i=1}^{N} v_i(t)^2/N$ and $W(t) = [E(t_2) - E(t_1)]/(t_2 - t_1)$ where t_2 and t_1 represent the times at which two successive collisions take place, are not independent from the total number of particles, but decay exponentially as $\sim \exp(-cN)$, showing that the system does not possess a proper thermodynamic limit. The existence of thermodynamic limit in real granular systems is not clear, but it seems a rather natural requirement in a statistical mechanical approach.

Because of the inelastic collisions, in order to reach a statistically stationary situation some energy must be injected into the system. This is achieved in our model by the random noise term acting on each particle. This term mimics the action of a vibrating box. Notice that the boundary conditions and the energy-pumping mechanism are different from that of Ref. [9] and present the advantage of providing a "good" thermodynamic limit as far as the energy E and energy dissipation W are concerned, i.e., $\langle E \rangle$ and $\langle W \rangle$ become independent of N for large values of N.

Moreover, our system does not have walls and the clustering is nontrivial. On the other hand, the energy feeding mechanism adopted in [7] forces one to introduce a somehow artificial cooling of the particles, by renormalizing the velocity of the center of mass at each time step. In our formulation this procedure is overcome by the presence of the thermal bath.

For each given choice of r and τ the system, after a long transient, reaches a stationary state with certain properties. The presence of two time scales, namely τ and the mean collision time τ_c, leads to different dynamical regimes.

(a) When $\tau \ll \tau_c$ it is easy to argue that the grains reach a rather simple statistical equilibrium and that their velocity distribution is that of an ideal gas with an effective temperature T_F^*, slightly lower than the temperature of the "heat bath," T_F.

(b) In the opposite limit $\tau \gg \tau_c$ the driving mechanism towards the macroscopic stationary state is dominated by the collision process itself.

Two phenomena are observed:

(1) the velocity distribution ceases to be Gaussian and the deviation becomes more and more pronounced with decreasing values of the restitution coefficient r.

(2) The spatial distribution becomes strongly inhomogeneous.

It is worth stressing that, at variance with the clusterization in Ref. [9], in our case the clusters are created and destroyed continuously in the system as long as the system evolves. The inhomogeneity in the spatial distribution of the grains, see Fig. 1, can be quantitatively characterized by the so-called Grassberger-Procaccia dimension d_2 that we compute from the correlation function $C(R)$ defined as

$$C(R) = \frac{1}{N^2}\frac{1}{T}\int_0^T dt \sum_{i<j} \theta[R - |x(t)_i - x_j(t)|] \sim R^{d_2}, \tag{3}$$

where T represents the duration of the simulation. In Fig. 2 we show $C(R)$ vs R for a clustering situation. It turns out that d_2 is lower than 1 when $r < 1$, e.g., for $\tau = 100$ and $r = 0.6$, $d_2 = 0.59$, while for $\tau = 100$ and $r = 0.9$ $d_2 \simeq 1$.

Notice that in this case the stationary regime is brought about by the collisions and that these occur more frequently in the regions of higher density. For instance, for $\tau = 100$ and $r = 0.7$ one gets for the number of collisions as a function of the spatial density ρ, $N_{\mathrm{coll}}(\rho) \sim \rho^2$.

Assuming that in the stationary state the power dissipated through the collisions must balance the power adsorbed from the heat bath, one derives the following relation between the temperature T_F, the average energy E, the power dissipated by the collisions W, and τ [13]:

$$W = \frac{1}{\tau}(T_F - 2E). \tag{4}$$

Notice that in the absence of collisions (or $\tau \ll \tau_c$) $T_F = 2E$ and $W = 0$ as in the ideal gas. Equation (4) is well satisfied for different values of τ and r.

As we quoted already in the inelastic regime one observes a strong deviation from the Gaussian behavior for the velocity distribution. In Fig. 3 we display the velocity distribution in a nearly elastic case ($\tau = 0.01$ and $r = 0.99$) and in a strong inelastic regime ($\tau = 100$ and $r = 0.7$). As it is possible to see it exists an evident departure from the Gaussian behavior in the inelastic case, where the velocity distribution shows almost exponential tails. The above result is close to that observed in Refs. [8,12].

Let us now try to relate the clustering properties of the system to the velocity distribution. In order to do that we consider the following quantities: the distribution of boxes, $N_{\mathrm{box}}(m)$, containing a given number, m, of

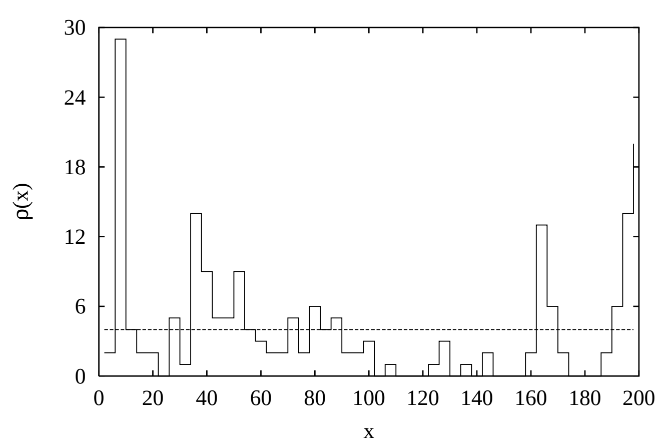

FIG. 1. Snapshot of the particle density at a given time for $T_F = 1$, $r = 0.6$, $\tau = 100$, $N = 200$, $L = 200$. All quantities are in arbitrary units. The dashed line represents the homogeneous density.

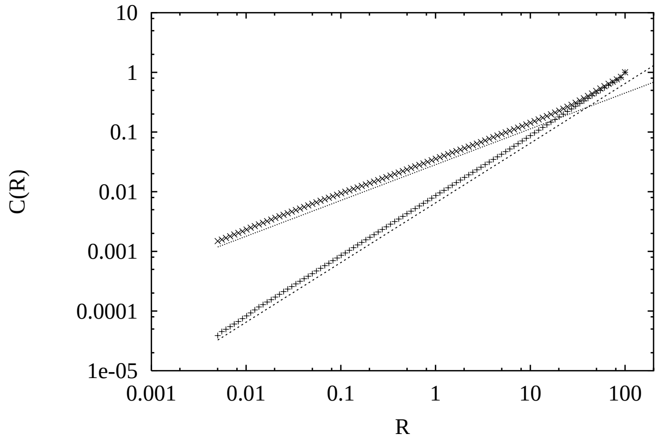

FIG. 2. $C(R)$ against R for two different choices of parameters: $\tau = 100$, $r = 0.6$ (top) and $\tau = 100$, $r = 0.99$ (bottom). In both cases $T_F = 1$, $N = 200$, and $L = 200$. The dimension takes, respectively, the values $d_2 = 0.59$ and $d_2 = 1$.

particles and the average kinetic energy, $E_{\rm kin}(m)$, in a box occupied by m particles [14]. Making the hypothesis that in each box the average velocity of the particles is zero, i.e. $\langle v(m)\rangle = 0$ (very well confirmed by the numerical data), one finds that $E_{\rm kin}(m)$ provides a measure of the variance of the velocity distribution in each box: $E_{\rm kin}(m) \simeq \frac{1}{2}\langle v^2(m)\rangle = \frac{1}{2}\sigma^2(m)$. We consider first the nonclusterized case ($\tau \ll 1$ and $r \simeq 1$). Within this regime we find from the simulations that:

$$\sigma^2_{\rm elas}(m) \simeq \text{const}, \qquad N^{\rm elas}_{\rm box}(m) = \frac{\lambda^m e^{-m}}{m!}, \tag{5}$$

where $\lambda = N/N_{\rm discr}$ is the average number of particles in each box and $N^{\rm elas}_{\rm box}(m)$ is a Poisson distribution. By assuming in each box a Gaussian velocity distribution with a constant variance $\sigma^2_{\rm elas}(m)$ it turns out that the global velocity distribution $P_{\rm elas}(v)$ is Gaussian. Let us recall that the Poisson distribution is the one associated with a process of putting independently λN particles into N boxes.

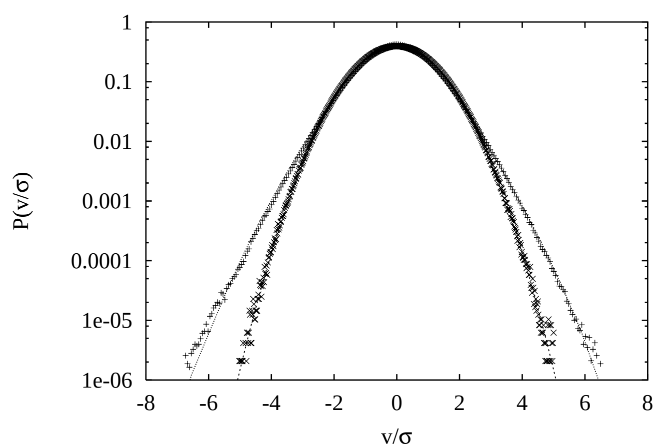

FIG. 3. Rescaled velocity distribution $P(v/\sigma)$ against v/σ: crosses are simulation data with $\tau = 100$, $r = 0.7$, the dashed line represents toy-model data, $\times$'s are simulation data with $\tau = 0.01$ and $r = 0.99$, and the dot-dashed line represents the Gaussian distribution. In both cases $T_F = 1$, $N = 200$, and $L = 200$.

Let us turn to the nonelastic case. If $\tau = 100$ and $r = 0.7$, considering the occupied boxes ($m > 0$), we obtain from the simulations:

$$\sigma^2_{\rm inel}(m) \sim m^{-\beta}, \qquad N^{\rm inel}_{\rm box}(m) = \frac{e^{-\alpha m}}{m}, \tag{6}$$

with $\beta \simeq 0.5$ and $\alpha \simeq 0.14$. Let us compute from these scalings the global velocity distribution. Taking into account that the spatial probability distribution of the particles is $N^{\rm inel}_{\rm box}(m)$ and assuming that their local velocity distribution is Gaussian, but with a variance $\sigma^2_{\rm inel}(m) \simeq m^{-\beta}$ which depends on the occupancy, we obtain, for the global velocity distribution $P_{\rm inel}(v)$, which in the continuum limit should correspond to

$$P_{\rm inel}(v) \simeq \sum_{m=1}^{\infty} e^{(-v^2 m^\beta/2)} e^{-\alpha m}. \tag{7}$$

We stress how it exists an astonishing agreement between the numerical results and the ones obtained with a toy model which just makes the following hypothesis: (i) Non-Poissonian distribution for the box occupancy; (ii) Gaussian distribution of velocities in each box with a density-dependent variance.

The hypothesis about the scaling relation between the velocity variance and the local density, apart from being justified numerically, can be understood in the following way. The stationarity and the scale invariance of the cluster distribution, implies a certain distribution of lifetimes for the clusters. In particular each cluster has a lifetime which is inversely proportional to its size. The scale-invariant cluster-size distribution thus implies a scale-invariant distribution for the lifetimes. The cluster lifetime (its stability) is strictly related to the variance of the velocity distribution inside the cluster itself. In order to ensure the stability of a cluster in a stationary state we have to require that the velocities of the particles belonging to it are not too different, or equivalently that the variance of the distribution is smaller the higher the density. So, given a scale-invariant distribution of clusters one would expect a scale-invariant distribution of variances (6). An independent check is provided by the behavior of the average relative velocities in boxes with different numbers of particles. For $\tau = 100$ and $r = 0.7$ one obtains

$$v^{\rm inel}_{\rm rel}(m) \sim m^{-\gamma}, \tag{8}$$

with $\gamma \simeq 0.3$, which indicates that the stability of a cluster is connected with the smallness of the velocity fluctuations inside it. In other words, the spatial clusterization corresponds to a clusterization in velocity space.

This analysis shows that in the present model it does exist a relation between the clustering phenomenon (in physical space or in velocity space) and the non-Gaussian distribution of the velocities.

A natural question to ask is whether our findings can be an artifact of the one dimensional dynamics. Can one expect to observe the non-Maxwell distribution and the clusterization even in higher dimensions? In order to clarify such an issue, we consider a different stochastic process. All the particles perform the Brownian motion described by Eq. (1) if $t \epsilon [K\Delta t, (K + 1)\Delta t]$ (where K is an integer number), while at the instants $t_K = K\Delta t$ each particle may collide with probability p with one of those particles which are spatially close to it according to the Boltzmann collision-number ansatz (BS).

In practice we perform the following algorithm *à la* Bird [15]: at each discrete time t_K for each particle, i, we extract out of a uniform distribution in the interval $[0, 1]$ a random number y. If $y > p$ there is no collision, otherwise the particle i scatters with another particle j if $|x_i(t_K) - x_j(t_K)| < l$, and their collision probability is proportional to $|v_i(t_K) - v_j(t_K)|$. This process renders the BS approximation exact; in fact in the limit $N \to \infty$, $p \to 0$, $l \to 0$, $\Delta t \to 0$ one can write for the stochastic process described above the following Boltzmann equation for the one particle distribution $P(x, v, t)$ [16]:

$$\frac{\partial}{\partial t} P + \frac{\partial}{\partial x}(vP) - \frac{1}{\tau}\frac{\partial}{\partial v}(vP) - \frac{T_F}{\tau}\frac{\partial^2}{\partial v^2} P = \frac{\partial}{\partial t} P|_{\mathrm{coll}} \tag{9}$$

$$\frac{\partial}{\partial t} P|_{\mathrm{coll}} = \frac{4\Lambda}{(1 + r)^2}\int dv' P(x, v', t) P\left(x, \frac{(2v - (1 - r)v')}{(1 + r)}, t\right)|v' - v| - \Lambda \int dv' P(x, v', t) P(x, v, t)|v' - v|, \tag{10}$$

where $\Lambda \sim p/\Delta t \sim 1/\tau_c$ and depends on the mean particle density.

It is easy to show that in the limit of elastic collisions $(r = 1)$ the velocity distribution becomes Gaussian for any value of τ, Δt, and p. On the contrary, for $r \neq 1$ an analytical solution is not known.

In order to further clarify the relevance of the dimensionality, we have performed some simulations in 2D. In this case we find non-Gaussian tails for the velocity distribution, together with clusterization, at large values of τ and strong inelasticity. In 2D the velocities in the collision change according to the following rule:

$$v_i(t_K + 0^+) - v_j(t_K + 0^+) = r\hat{e}[v_i(t_K) - v_j(t_K)], \tag{11}$$

where $\hat{e}$ is a two-dimensional unit vector of random orientation. In this case the results are qualitatively similar to the ones obtained in one dimension, i.e., at large values of τ and strong inelasticity the Grassberger-Procaccia dimension is smaller than 2 and the velocity distribution is not Gaussian.

In summary, we introduced a model of granular gas with inelastic collisions between the particles. The system exhibits a variety of regimes ranging from a completely elastic case without clusterization and Gaussian distribution for the velocities to an inelastic regime with strong clusterization and non-Gaussian velocity distribution. In this framework we have shown a possible scenario to relate the clustering properties of the system to the velocity distributions. These results seem promising and give us the hope to be able to perform analytic work based on the Boltzmann approach in order to clarify further the model.

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