Soliton Creation and Destruction, Resonant Interactions, and Inelastic Collisions in Shallow Water Waves

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Shallow water waves are studied using a nonlinear wave equation (W2) derived from Euler's equations by Whitham's method: W2 is the Korteweg–de Vries (KdV) equation plus higher-order correction terms. By projecting numerical simulations of W2 onto the soliton and radiation modes of the inverse scattering transform for the KdV equation we (i) generalize the soliton concept to higher order, (ii) provide a rigorous interpretation of a new soliton resonance effect, (iii) demonstrate that solitons and radiation undergo inelastic collisions, and (iv) find evidence for soliton creation and destruction. [S0031-9007(98)07500-0]

The study of shallow water waves in $1 + 1$ dimensions has often focused on the Korteweg–de Vries (KdV) equation $u_t + 6uu_x + u_{xxx} = 0$, for which solitons were discovered by Zabusky and Kruskal [1] (ZK). These results subsequently led to the discovery of the inverse scattering transform (IST), a method for solving the *Cauchy problem* for soliton equations [2,3]. To investigate higher-order wave behavior one can use the method of Whitham [4] (see also Taniuti [5]) to carry out a multiscale expansion of the Euler equations in shallow water. At 1 order of approximation higher than the KdV equation one finds in normalized form

$$u_t + \varepsilon(6uu_x + u_{xxx}) + \varepsilon^2(\alpha_1 u_{5x} +$$

$$\alpha_2 uu_{xxx} + \alpha_3 u_x u_{xx} + \alpha_4 u^2 u_x) = O(\varepsilon^3). \quad (1)$$

The constant coefficients are given by $\alpha_1 = 19/10$, $\alpha_2 = 10$, $\alpha_3 = 23$, and $\alpha_4 = -6$. We refer to (1) as the second Whitham equation (W2), which is valid for small values of the dimensionless parameters $\alpha = a/h$ and $\beta = (h/l)^2$, for a the wave amplitude, l the wave length, and h the water depth. In (1) one assumes that $\varepsilon \sim \alpha \sim \beta$ in the multiscale expansion. The simple transformation $x \rightarrow \sqrt{\varepsilon}\, x$, $t \rightarrow \sqrt{\varepsilon}\, t$, and $u \rightarrow u/\varepsilon$ renders (1) (and our numerical study) independent of the expansion parameter ε. W2 has the same form as the second Lax equation [6] (L2) in the infinite hierarchy of higher KdV flows; L2 is integrable by IST and is identical to (1) except that it has different coefficient values, $\alpha_1 = 1$, $\alpha_2 = 10$, $\alpha_3 = 20$, and $\alpha_4 = 30$. It is generally thought that the particular coefficients in the *physical equation* W2 render (1) nonintegrable [7]. Asymptotic integrability $[O(\varepsilon^2)]$ has been established by showing that W2 can be derived (a) from the second Lax equation [6] (L2) by a *Lie transformation* [7] and (b) from the KdV equation by a *master symmetry transformation* [8]. In the present paper we numerically study the physical behavior and possible (near) integrability of Eq. (1).

We study W2 in much the same way that ZK studied the KdV equation originally, i.e., by carrying out numerical experiments with periodic boundary conditions for

sine wave initial conditions. We have developed a Fourier spectral code for numerically integrating KdV, L2, and Eq. (1) [9]. In order to check out the code we used the specific coefficient values for both KdV and L2 and projected the numerical solutions onto the IST modes. These modes are known theoretically to be constants of the motion for KdV and L2 evolutions [6] and were also found by us to be numerically constant to high precision.

In the numerical study for W2 we use 1024 spatial points with time discretization $\Delta t = 2 \times 10^{-5}$, and we graph space series every $\Delta t = 5 \times 10^4$ time steps (establishing the arbitrary time unit $t = 1$). We have $\alpha = 0.12$ and $\beta = 2.8 \times 10^{-4}$ for the initial sine wave. Our motivation for studying this particular case is that it reduces the number of solitons from nine [1,10] to five which allows us to observe the solitons while simultaneously visualizing other nonlinear effects which we have found to be important. Figure 1 gives the results of the numerical integration for the KdV equation: In panel (a) we show the space-time evolution of the wave amplitude starting from the initial sine wave at time $t = 0$. Panel (b) shows the colored contours of this evolution. Five solitons are seen to emerge from the sine wave initial condition and to undergo integrable, nonlinear dynamical motions, including interaction phase shifts [1].

In Fig. 2 we show the evolution of the W2 equation starting from the same sine wave initial condition. While there are five "solitons" which occur in the higher-order dynamics, we note that the evolution is considerably different than that for the KdV equation, as can be seen by comparing the respective (a) panels of Figs. 1 and 2 [compare also (b) panels]. For times prior to $t \sim 40$ both KdV and W2 evolutions are quite similar. For the W2 equation, for $t \overset{>}{\sim} 45$, the evolution changes dramatically from that described by the KdV equation: A row of very regular oscillatory waves appears whose crests are parallel to the time axis (they have zero phase speed) and which run from about 64 to 224 along the space axis. The oscillatory component is then subsequently "mixed" and "dragged" rather uniformly over the *x-t* domain
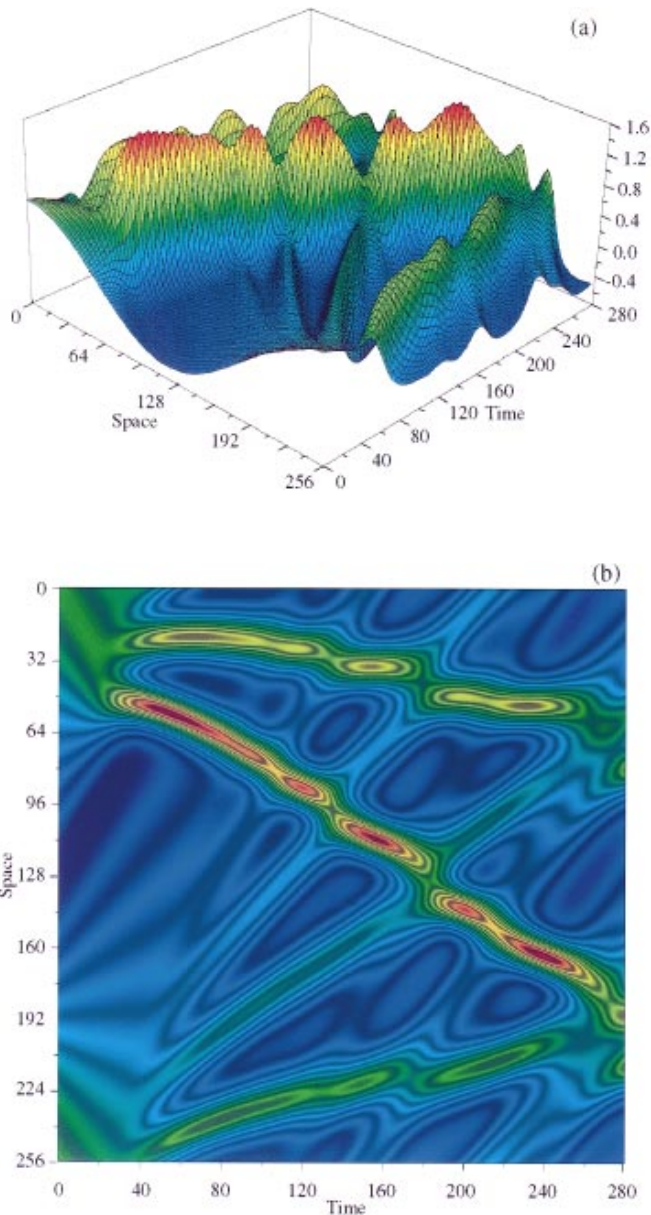
FIG. 1(color). (a) Space-time evolution of the KdV equation for a sine wave initial condition. (b) Smoothed contours of the dynamics shown in panel (a).

In order to better understand how the nonlinear soliton resonance occurs we show in Fig. 3 the *space series* of the evolution of the KdV and W2 equations at $t = 45$, which is the moment in time when we estimate that the resonantly produced wave packet has reached its maximum size. The evolution of both the KdV and W2 equations appears quite "normal" for earlier times (i.e., "KdV-like"), but near $t \sim 45$ the W2 evolution resonantly emits the packet of oscillatory waves which is seen to emerge at the right-hand, leading edge of the largest soliton.

Up to the present point we have investigated the evolution of the W2 wave trains in *configuration space*. We now use the inverse scattering transform of the KdV
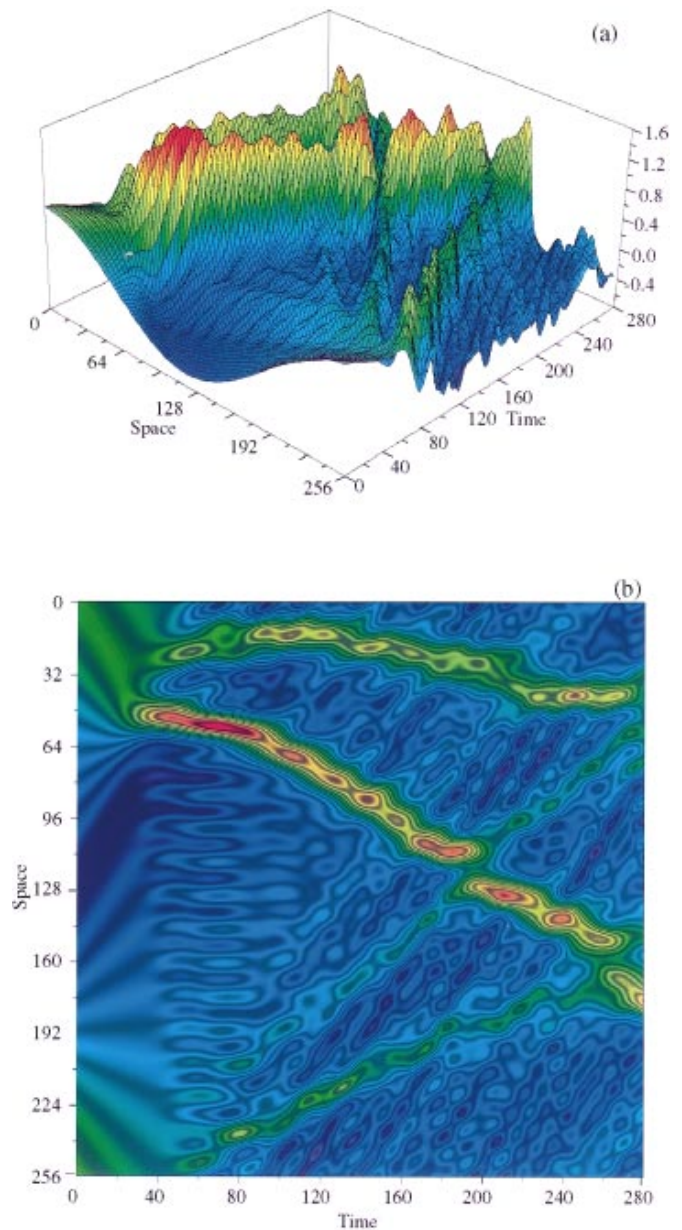

FIG. 2(color). (a) Space-time evolution of the W2 equation for a sine wave initial condition. (b) Smoothed contours of the dynamics shown in panel (a).

by the solitonic component. The numerical simulations suggest that there is a "nonlinear resonant excitation" of the oscillatory modes which is effectively initiated and "driven" by the solitons for a brief interval of time near $t \sim 45$. Thereafter the W2 dynamics consists of the five soliton modes evolving and interacting in the background field of oscillatory waves (Fig. 2). Note that the expression "nonlinear resonance" as used herein refers to the *resonant excitation of oscillatory modes by solitons*. Thus the present resonant behavior occurs at 1 (singular perturbative) order of approximation higher than classical resonances in which linear sine waves exchange energy: In the present study it is the solitons which undergo energy exchange.
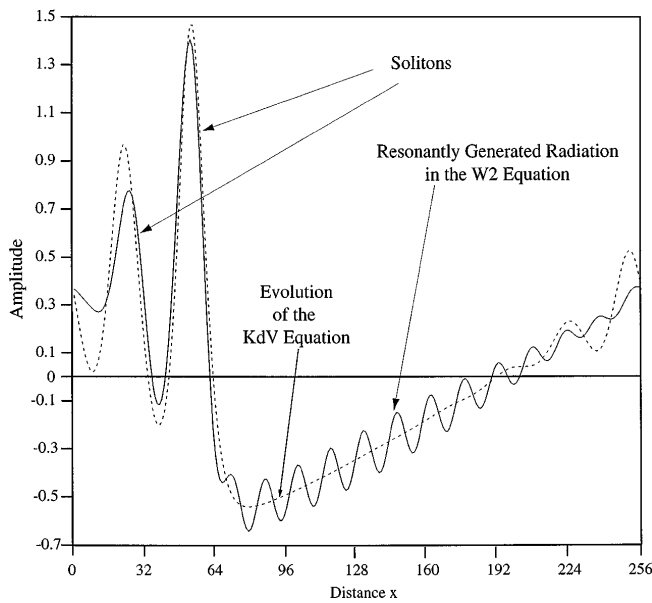
FIG. 3. Comparison between the nonlinear evolution of the KdV equation (dotted line) and the W2 equation (solid line) for $t = 45$. While the KdV equation dynamically evolves five solitons in classical fashion, the W2 equation instead initiates a soliton/radiation resonance which injects energy into a wave packet lying to the immediate right of the largest soliton.

equation to analyze the numerical results in *wave number space* [11,12]. For the periodic boundary conditions used here the IST modes are *cnoidal waves* (the Jacobian elliptic function *cn*) which have the form of *sine waves* when their *moduli* are small with respect to 1 and are *solitons* for moduli near 1. The results are given in Fig. 4(a) for KdV evolution, where we show the IST spectrum in wave-number space as it evolves in time. The actual spectral components consist of five *soliton modes* at low wave number, while the nearby components are *small amplitude radiation* (modes 6–8). Mode numbers 9 and above do not significantly contribute to the dynamics because they have very small amplitudes. We see that all spectral components are constants of the motion for the KdV equation during the entire period of the simulations [total time $T \sim 880$, or about one Fermi-Pasta-Ulam (FPU) recurrence time]. These latter results are gratifying because the IST mode amplitudes are theoretically constants of the motion for pure KdV dynamics: Fig. 4(a) thus demonstrates the accuracy of the numerical algorithm used to integrate the equation of motion: The numerical code conserves the IST spectral amplitudes for at least one FPU recurrence period. We have also checked L2 evolution for the same sine wave initial condition; we find the modes to be identical to those of Fig. 4(a), as theoretically required [6].

In Fig. 4(b) we show the time evolution of the IST spectrum for the W2 equation. Note that in this case *the IST modes for W2 are not constants of the motion*. This is of course to be anticipated, since we are projecting the motion of the W2 equation (which we presume not to

be integrable) onto the IST modes for the KdV equation (which is integrable). We conduct our IST analysis in the hope that W2 dynamics will, in some sense, be a relatively small perturbation of KdV dynamics, at least to the extent that *any time variation in the projection of the W2 dynamics onto the IST modes will be sufficiently slow* so that useful physical information may be obtained. This turns out to be the case as we now discuss.

Note in Fig. 4(b) that the IST modes for the W2 equation are relatively constant up to the early time $t \sim 40$; during this short time interval the W2 spectrum is seen to remain similar to that for the spectrum of the KdV equation as shown in Fig. 4(a). Times $t \stackrel{>}{\sim} 45$ are characterized simultaneously by (a) a rapid, momentary reduction in the amplitudes of the two largest solitons [Fig. 4(b)] and (b) a rapid growth in radiation modes near wave number
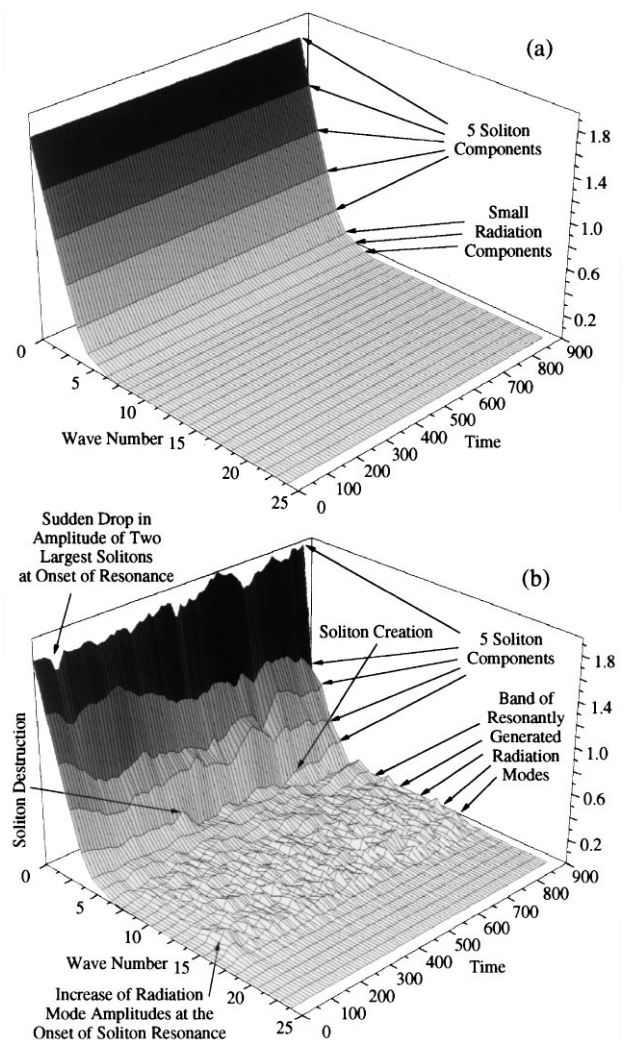

FIG. 4. (a) Inverse scattering transform for the evolution of the KdV equation. Note that the nonlinear mode amplitudes (as a function of wave number) are constants of the motion for the entire simulation time, $t = 880$. (b) Inverse scattering transform for the evolution of the W2 equation. Each nonlinear mode, in contrast to that for the KdV equation, is seen to have rather complex wave-number/time dynamics.

~16. Continued nonlinear evolution smears out the radiation components over a band of wave numbers $\sim 8-18$, for which the IST mode amplitudes are never constant during the remainder of the evolution. Nevertheless, these collective radiation modes maintain a rather energetic presence over this range of wave numbers.

In Fig. 4(b) we see that there is considerable variation in the soliton amplitudes throughout the evolution of the W2 equation. Basically, each soliton-soliton interaction results in a relatively small change in soliton amplitude (and energy), indicating that solitons in the W2 equation do not preserve their identities and phase speeds after a collision, as is the case for the KdV equation. We therefore interpret the W2 soliton-soliton interactions to be somewhat *inelastic* although not drastically so. Furthermore, it is clear that the soliton-radiation interactions are themselves inelastic, since the radiation mode amplitudes continue to vary slowly throughout their evolution. However, the band of excited radiation modes is relatively persevering as can be seen in Fig. 4(b). While the largest soliton is found to oscillate erratically in amplitude as a function of time, it also *slowly grows in amplitude* over the simulation period, evidently at the expense of its two nearest-neighbor solitons whose amplitudes also oscillate in time while *slowly decreasing*.

We have also checked the moduli of the IST spectral (cnoidal wave) components to see how they vary in time. Does a soliton at $t = 0$ remain a soliton for all future times in W2 evolution? This does not necessarily happen, as we see in the present case, since the smaller of the five solitons loses so much energy near $t \sim 350$ that, for a period of about 300 time units, it ceases to be a soliton, i.e., its modulus falls substantially below ~ 0.99 during this time interval [see Fig. 4(b)]. At a later time it becomes re-energized (by inelastic collisions with other solitons and radiation) and subsequently becomes a soliton again: The modulus, after decreasing to ~ 0.2 at the beginning of this period, later increases to ~ 0.99 for times >650. We suspect this to be a rather generic effect in higher-order nonlinear systems. In fact if we had started our simulation at $t \sim 400$ (using the appropriate wave form at that moment in time) we would have concluded that the fifth W2 projected mode in the IST spectrum was *not initially a soliton*, but at a later time it evolved into a soliton thanks to the nonlinear inelastic interactions. Thus both *soliton creation and destruction* are possible outcomes of interactions in the W2 equation.

The higher-order dynamics studied here are found to be quite rich in their complexity, although nonintegrable. It is tempting to conclude that the results might be interpreted as the "KdV equation plus a small perturbation." We have resisted this interpretation for a number of reasons, primarily because some effects are clearly *not* perturbative over long time scales, e.g., soliton creation and destruction. Based upon our numerical results the long time evolution of (1) probably consists of only one large soliton plus background "radiation," hardly a perturbative

result. Perturbation theory for infinite-line boundary conditions [8] must give way to periodic boundary conditions in the present case. The periodic theory for the KdV equation [2,3,11,13] is quite technical in that it requires the implementation of Riemann theta functions for constructing spectral solutions. Floquet theory and Block eigenfunctions are necessary for the *direct IST spectral problem* and the construction of the theta function parameters using (a) physical effectivization [11], (b) algebraic geometry [3,13], or (c) the theory of Schottky groups [3] are highly nontrivial for the general *genus N* case. Perturbation theory is therefore clearly not an easy task, but we hope that subsequent theoretical work will be stimulated by our results.

[1] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).

[2] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).

[3] V. E. Zakarov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii, *Soliton Theory-Method of The Inverse Scattering Problem* (Nauka, Moscow, 1980); M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981); R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, London, 1982); A. C. Newell, *Solitons in Mathematics and Physics* (SIAM, Philadelphia, 1985); P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University Press, Cambridge, 1989); E. D. Belokolos, A. E. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Springer-Verlag, Berlin, 1994).

[4] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).

[5] T. Taniuti, Suppl. Prog. Theor. Phys. **55**, 1 (1974).

[6] P. D. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).

[7] Y. Kodama, Phys. Lett. **112A**, 193 (1985); **112A**, 245 (1985).

[8] A. S. Fokas and Q. M. Liu, Phys. Rev. Lett. **77**, 2347 (1996).

[9] B. Fornberg and G. B. Whitham, Philos. Trans. R. Soc. London A **289**, 373 (1978).

[10] A. R. Osborne and L. Bergamasco, Physica (Amsterdam) **18D**, 26 (1986).

[11] J. P. Boyd, J. Math. Phys. (N.Y.) **25**, 3390 (1984); **25**, 3402 (1984); A. R. Bishop, M. G. Forest, D. W. McLaughlin, and E. A. Overman II, Physica (Amsterdam) **18D**, 293 (1986); A. R. Bishop and P. S. Lomdahl, Physica (Amsterdam) **18D**, 26 (1986).

[12] A. R. Osborne, Phys. Rev. E **52**, 1105 (1995).

[13] S. P. Novikov, S. V. Manakov, L. P. Pitayevsky, and V. E. Zakharov, *Theory of Solitons. The Inverse Scattering Method* (Consultants Bureau, New York, 1984).