

Universality of Transport Properties in Equilibrium, the Goldstone Theorem, and Chiral Anomaly

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We study transport in a class of physical systems possessing two conserved chiral charges. We describe a relation between universality of transport properties of such systems and the chiral anomaly. We show that the nonvanishing of a current expectation value implies the presence of gapless modes, in analogy to the Goldstone theorem. Our main tool is a new formula expressing currents in terms of anomalous commutators. Universality of conductance arises as a natural consequence of the nonrenormalization of anomalies. To illustrate our formalism we examine transport properties of a quantum wire in 1 + 1 dimensions and of massless QED in a background magnetic field in 3 + 1 dimensions. [S0031-9007(98)07366-9]

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Transport properties of a physical system are usually linked to complicated dynamical processes (such as impurity scattering, interparticle interactions, etc.) and, in general, are not universal. Systems not exhibiting any dissipative processes may, however, exhibit some universal transport coefficients that are insensitive to changes in the microscopic constitution of the system. When one encounters a universal transport coefficient one should look for a physical principle explaining its universality. In the quantum Hall effect, for instance, the universality of the Hall conductance can be linked to gauge invariance [1]. Another example is superconductivity, where it is the spontaneous breaking of the U(1) gauge symmetry [2] that leads to the vanishing of the longitudinal resistivity and to the Meissner effect. In both examples the conductivity is universal with amazing accuracy, which is the result of the existence of a gap in the spectrum of bulk charged excitations. Note, however, that an incompressible quantum Hall system with an edge must support a branch of compressible edge states, which play an essential role in understanding Hall quantization.

In this paper we consider a class of physical systems having no gap for charged excitations, yet exhibiting universal transport properties. At low energies these systems are assumed to possess two *commuting conserved chiral charges* Q_L and Q_R corresponding to the particles of left and right chirality. Although our method is not limited to the electric transport, we consider the electric conductance as a representative example. The latter is defined as $G = I/V$, where I is the electric current and $eV = \mu_L - \mu_R$ is the difference between the chemical potentials μ_L and μ_R of the reservoirs of the particles of left and right chirality. We show that the conductance G is independent of the dynamics of the system as long as the charges Q_L and Q_R are conserved. The principle that protects the universality is the nonrenormalization of chiral anomalies by interactions [3]. To illustrate our

conclusions, we examine the transport properties of a quantum wire in 1 + 1 dimensions and of massless QED in a background magnetic field in 3 + 1 dimensions.

The main tool we are using is a new formula (5) which relates the dc current to the anomalous commutators. In order to obtain this formula we use the methods of *equilibrium* statistical mechanics—an approach that has proven to be effective at the description of some transport phenomena in solids [4–6]. The only condition imposed on the dynamics of the system is the existence of two *commuting conserved charges* Q_L and Q_R ,

$$[\mathcal{H}, Q_L] = [\mathcal{H}, Q_R] = 0, \quad [Q_L, Q_R] = 0. \quad (1)$$

Here \mathcal{H} is the Hamiltonian of the theory. We denote the conserved Noether currents corresponding to the charges Q_L and Q_R by j_L^μ and j_R^μ . In the examples below, j_L and j_R are chiral currents corresponding to the fermions of left and right chirality (in 1 + 1 or in 3 + 1 dimensions). The observable we are interested in is the electric current $j_e^\mu = e(j_L^\mu + j_R^\mu)$ (e is the elementary electric charge). Henceforth, we will refer to the difference $j_a^\mu = e(j_L^\mu - j_R^\mu)$ as the axial current.

Physically, the conservation of the charges Q_L and Q_R means that there is no scattering of the particles of left chirality to the right ones and vice versa.

The conserved charges Q_L and Q_R are conjugate to the chemical potentials μ_L and μ_R of the external reservoirs of the particles of left and right chirality. The thermal state of the system connected to the external reservoirs is given by the density matrix,

$$\Sigma_\mu = e^{-\beta \mathcal{H}_\mu}, \quad \mathcal{H}_\mu = \mathcal{H} + \mu_L Q_L + \mu_R Q_R, \quad (2)$$

and its transport properties are described by *equilibrium* statistical mechanics.

The continuity equation for the electric current $j_e^\mu = (\rho, \mathbf{j})$ reads $\partial_\mu j_e^\mu = 0$. In $d + 1$ dimensions it can be

solved in terms of an antisymmetric tensor field b of rank $(d-1)$, $j_e^\mu = \epsilon^{\mu\nu_1\dots\nu_d} \partial_{\nu_1} b_{\nu_2\dots\nu_d}$, where $\epsilon^{\mu\nu_1\dots\nu_d}$ is Levi-Civita's antisymmetric tensor. Note that the field b is not a physical observable of the system. A shift $b \mapsto b + h$, where the tensor h satisfies $\epsilon^{\mu\nu_1\dots\nu_d} \partial_{\nu_1} h_{\nu_2\dots\nu_d} = 0$, does not change the physical quantities j^μ and can be interpreted as a gauge transformation. This gauge freedom results in the field b having only d physical degrees of freedom. One may choose a specific gauge for the potential b which allows one to explicitly express the physical degrees of freedom of the field b in terms of a d -vector field \mathbf{a} :

$$\rho = e\nabla \cdot \mathbf{a}, \quad \mathbf{j} = -e\partial_t \mathbf{a}. \quad (3)$$

In the equilibrium state characterized by the chemical potentials μ_L and μ_R , the expectation value of the current is given by

$$\begin{aligned} \langle \mathbf{j}(x) \rangle_\mu &= -e \langle \partial_t \mathbf{a}(x) \rangle_\mu \\ &= \frac{ie}{\hbar} \langle [\mathcal{H}, \mathbf{a}(x)] \rangle_\mu \\ &= \frac{ie}{\hbar} Z_\mu^{-1} \text{Tr}\{e^{-\beta H_\mu} [\mathcal{H}_\mu, \mathbf{a}(x)]\} \\ &\quad - \frac{ie}{\hbar} Z_\mu^{-1} \text{Tr}\{e^{-\beta H_\mu} [\mu_L Q_L + \mu_R Q_R, \mathbf{a}(x)]\}, \end{aligned} \quad (4)$$

where $Z_\mu = \text{Tr}\Sigma_\mu$. Note that, in the quantum theory of thermal equilibrium states (see, e.g., [7]), imaginary-time Green functions in the grand-canonical ensemble are constructed in terms of the "Hamiltonian" \mathcal{H}_μ . However, the physical time evolution of operators in the Heisenberg picture is determined by the Hamiltonian \mathcal{H} . For operators which commute with conserved charges, the time evolutions determined by \mathcal{H}_μ and by \mathcal{H} coincide. But for general operators, such as creation and annihilation operators, the true time evolution is the one determined by \mathcal{H} .

At first sight both terms on the right-hand side (rhs) of (4) must vanish by cyclicity of the trace, because \mathcal{H}_μ and $\mu_L Q_L + \mu_R Q_R$ commute with $\exp(-\beta \mathcal{H}_\mu)$. However, we are not allowed to use the formula $\text{Tr}[a, b]c = \text{Tr}(abc) - \text{Tr}(bac)$, because the products of three operators define unbounded operators which are not of trace class [8]. This problem is a manifestation of zero-mode divergences.

A more careful analysis shows that only the first trace on the rhs of (4) vanishes. Indeed, if we regularize the system by modifying the Hamiltonian \mathcal{H} in such a way that a small spectral gap above the ground state energy is opened then, for a system in a finite box, $e^{-\beta \mathcal{H}_\mu}$ is of trace class, and the appropriately smeared field $\mathbf{a}(x)$ is bounded by some function of \mathcal{H}_μ . Then the first trace on the left-hand side (lhs) of (4) vanishes, by cyclicity of the trace. But this argument *cannot* be applied to the second

trace on the rhs of (4), because, after regularization, Q_L and Q_R do no longer commute with \mathcal{H}_μ .

Finally, formula (4) yields the universal result

$$\begin{aligned} \langle \mathbf{j}(x) \rangle_\mu &= -\frac{ie}{\hbar} \langle [\mu_L Q_L + \mu_R Q_R, \mathbf{a}(x)] \rangle_\mu \\ &= -\frac{i}{2\hbar} (\mu_L - \mu_R) \int d^d y \langle [j_a^0(y), \mathbf{a}(x)] \rangle_\mu. \end{aligned} \quad (5)$$

Formula (5) expresses the electric current in terms of the anomalous commutator of the time component of the axial current with the field \mathbf{a} solving the continuity equation.

Next, we want to show how nontrivial physical conclusions can be arrived at by applying formula (5) to concrete physical systems. Our first example is a one-dimensional interacting electron liquid (quantum wire). It had long been understood [4] that the conductance $G = I/V$ (where I is the electric current and V is the voltage drop) of a pure noninteracting quasi-one-dimensional electron system must be quantized in units of $2e^2/h$ [4]; i.e.,

$$G = 2n \frac{e^2}{h}, \quad n = 0, 1, 2, \dots, \quad (6)$$

where e is the elementary electric charge and h is Planck's constant. The factor of 2 on the rhs of (6) originates in the spin of electrons, the factor of n corresponds to the number of filled energy bands of transversal quantization which form one-dimensional conducting channels.

It was far from clear, however, how the electron-electron interaction influenced the conductance. For a long time it was believed that repulsive electron-electron interactions should suppress the conductance. Recent experiments [9] indicate that the suppression of conductance cannot be caused by electron-electron interactions alone, but is due to the presence of disorder coupling left- to right-movers and depends on the disorder potential and on temperature.

It was argued in [10] that the nonrenormalization of conductance by electron-electron interactions was due to the strong influence of the boundary conditions imposed by the reservoirs. This idea was supported by calculating the current-current correlation function of a model one-dimensional system where the reservoirs were modeled by turning off electron-electron interactions outside some finite region of the system. Conductance quantization in quantum wires and quantum Hall systems has been compared in [6].

We shall show that a pure quantum wire (with no impurity backscattering) satisfies the dynamical requirement (1) and the universality of conductance quantization, as expressed in Eq. (6), follows directly from formula (5). For simplicity, we shall consider spinless fermions and drop the factor of 2 in (6).

The Hamiltonian of a general one-dimensional interacting fermionic system is given by

$$\mathcal{H} = i\hbar v_F \int dx (\psi_L^* \partial_x \psi_L - \psi_R^* \partial_x \psi_R) + \mathcal{H}_{\text{int}}, \quad (7)$$

where ψ_L and ψ_R are left- and right-moving electrons of the noninteracting model, and \mathcal{H}_{int} is the interaction Hamiltonian. It includes higher order terms in ψ corresponding to electron-electron scattering as well as quadratic terms responsible for nonlinearity of dispersion. We refer to modes created by ψ_L^* and ψ_R^* as left- and right-movers in spite of the fact that dynamically they are not necessarily quasiparticles of the interacting model. One can introduce densities of left- and right-movers $\psi_L^* \psi_L = n_L$, $\psi_R^* \psi_R = n_R$. The total charge density is given by $\rho = e(n_L + n_R)$. The expression for the electric current density j in terms of ψ_L and ψ_R is not universal and depends on the particular form of \mathcal{H}_{int} .

If we assume that the junctions between the one-dimensional system and the electron reservoirs are adiabatic, the conserved charges conjugate to the chemical potentials of the reservoirs are equal to the integrals of n_L and n_R :

$$Q_L = \int dx n_L, \quad Q_R = \int dx n_R. \quad (8)$$

We assume that these charges commute with the interacting Hamiltonian (7).

It is convenient to use one-dimensional bosonization formulas [11] for the Fermi fields ψ_L and ψ_R ,

$$\psi_L = e^{-2\pi i \phi_L}, \quad \psi_R = e^{2\pi i \phi_R}. \quad (9)$$

The bosonic fields ϕ_L and ϕ_R satisfy the commutation relations

$$[\phi_{L,R}(x), \phi_{L,R}(y)] = \pm \frac{i}{4\pi} \varepsilon(x - y),$$

$$[\phi_L(x), \phi_R(y)] = \frac{i}{4\pi}, \quad (10)$$

where $\varepsilon(x - y) = 1, x > y$; and $\varepsilon(x - y) = -1, x < y$. The densities of left- and right-moving particles acquire the form $n_L = \partial_x \phi_L$, $n_R = \partial_x \phi_R$. The conserved charges Q_L and Q_R have the following commutation relations with the bosonic fields:

$$[Q_L, \phi_L(x)] = \frac{i}{2\pi}, \quad [Q_R, \phi_R(y)] = -\frac{i}{2\pi}. \quad (11)$$

The electric charge density is then given by

$$\rho = e(\partial_x \phi_L + \partial_x \phi_R) = e\partial_x a, \quad (12)$$

where a is the current potential $a = \phi_L + \phi_R$. We note that all of the commutation relations and bosonization rules listed above depend only on the kinematics of Fermi fields and are entirely independent of the dynamics of the system. Our only important dynamical assumption is the commutativity of the charges Q_L and Q_R with the Hamiltonian of the interacting system.

Formulas (5) and (11) can now be combined to yield the electrical conductance

$$\langle j(x) \rangle_\mu = -i \frac{e}{\hbar} \langle [\mu_L Q_L + \mu_R Q_R, a(x)] \rangle_\mu$$

$$= \frac{e}{\hbar} (\mu_L - \mu_R). \quad (13)$$

This concludes the derivation of the universal conductance formula (6).

Next, we test formula (5) on a (3 + 1)-dimensional example of massless Dirac fermions coupled to the electromagnetic field. This system is described by the Lagrangian

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \psi_L^* \sigma_L^\mu \left(i\hbar \partial_\mu - \frac{e}{c} A_\mu \right) \psi_L$$

$$+ \psi_R^* \sigma_R^\mu \left(i\hbar \partial_\mu - \frac{e}{c} A_\mu \right) \psi_R, \quad (14)$$

where $\sigma_\mu^L = (I, \sigma_k)$, $\sigma_\mu^R = (I, -\sigma_k)$. Chiral currents $j_L^\mu = \psi_L^* \sigma_\mu^L \psi_L$ and $j_R^\mu = \psi_R^* \sigma_\mu^R \psi_R$ are not conserved because of the chiral anomaly. The conservation is recovered upon adding a Chern-Simons term [12]:

$$\tilde{j}_{L,R}^\mu = j_{L,R}^\mu \pm \frac{\alpha^2}{8\pi^2 e^2} \epsilon^{\mu\nu\lambda\sigma} A_\nu \partial_\lambda A_\sigma, \quad (15)$$

where α is the fine structure constant $\alpha = e^2/\hbar c$. The corresponding charges,

$$Q_L = \int d^3x \tilde{j}_L^0, \quad Q_R = \int d^3x \tilde{j}_R^0, \quad (16)$$

are conserved. Note that the currents introduced in (15) are conserved, but fail to be gauge invariant. However, the charges Q_L and Q_R are not only conserved but also gauge invariant. Thus we may introduce the chemical potentials μ_L and μ_R canonically conjugate to these conserved charges. A situation where μ_L is different from μ_R is encountered in some models of the early Universe [13].

Our goal is to compute the expectation value of the electric current \mathbf{j} in the background electromagnetic field A_μ applying formula (5):

$$\langle \mathbf{j}(x) \rangle_A = -\frac{ie}{2\hbar} (\mu_L - \mu_R) \langle [Q_L - Q_R, \mathbf{a}(x)] \rangle_A. \quad (17)$$

The commutators of the densities of the left- and the right-handed fermions are given by [12]

$$[\tilde{j}_{L,R}^0(x), \tilde{j}_{L,R}^0(y)] = \pm \frac{i\alpha}{4\pi^2 e} \partial_k [B_k(x) \delta(x - y)], \quad (18)$$

whereas the commutator of the left-handed and the right-handed current is zero. Here $B_k = \epsilon^{ijk} \partial_i A_j$ is the magnetic field strength. The commutator of axial and electric charge densities is of the form

$$[\rho_a(x), \rho_e(y)] = \frac{ie\alpha}{2\pi^2} \partial_k [B_k(x) \delta(x - y)]. \quad (19)$$

Assuming that the commutator of ρ_a with \mathbf{a} is local, one can remove the divergence on the rhs of (19):

$$[\rho_a(x), a_k(y)] = \frac{i\alpha}{2\pi^2} B_k(x) \delta(x - y) + \dots, \quad (20)$$

where the ... stand for a term of the form of a curl of some vector field. Substituting (20) into (17) yields

$$\langle j_k(x) \rangle_A = \frac{\alpha}{4\pi^2 \hbar} (\mu_L - \mu_R) B_k(x). \quad (21)$$

Note that only the first term on the rhs of (20) contributes to the current. Another one drops out due to the integration in (16). The result (21) can be easily verified on the example of a noninteracting system in a constant uniform magnetic field, where the single-particle picture of [4] can be implemented. Our derivation implies that formula (21) holds true when the magnetic field is not necessarily uniform. In analogy to the previously considered quantum wire, the formula for the dc current (21) is not affected by interactions preserving charges Q_L and Q_R [3]. Coupling to the electromagnetic U(1) gauge field in (14) is an example of such an interaction.

Our next goal is to exhibit a relation between the conductance formula (5) and the Goldstone theorem. Recall that the Goldstone theorem states that in a system with spontaneous symmetry breaking there is a massless mode (Goldstone boson). At zero temperature the usual proof proceeds as follows (see, e.g., [14]): assume that the symmetry group is compact, and denote the symmetry generators by L^a , $L^a = \int d^d x j^a(x)$, where j^a stand for time components of conserved Noether currents. Spontaneous symmetry breaking manifests itself in a nonvanishing expectation value $\langle [L^a, \Phi(x)] \rangle \neq 0$ of a commutator of a symmetry generator L^a with some local operator Φ . Then,

$$\int d^d y \langle T[j^a(t, y)\Phi(s, x)] \rangle \Bigg|_{t=s-\epsilon}^{t=s+\epsilon} \neq 0. \quad (22)$$

Since the current j^a_ν is conserved, the integration surface can be deformed into a sphere of arbitrarily large radius R . In order to obtain a nonvanishing expectation value in (22), the correlation function of $j^a(y)$ and $\Phi(x)$ must decay as R^{-d} , where d is the spatial dimension. This implies the existence of massless modes in the system.

Note that the rhs of the universal conductance formula (5) is exactly of the form (22). The Noether current in this problem is the axial current $j_a(x)$; the symmetry group is the axial symmetry. Such as in the derivation of the Goldstone theorem, a nonvanishing expectation value of the current, $\langle \mathbf{j}(x) \rangle_\mu \neq 0$, implies the existence of a massless mode in the system. For instance, in the one-dimensional transport problem, this gapless mode is the density wave described by the field $a(x)$. The existence of this mode implies that, in the limit of large distance and low frequency scales, the system is described by a conformal field theory with a chiral algebra which contains a U(1) current algebra. In our simple example, this conformal field theory is described by the Luttinger model.

Although we observe a close analogy between the derivation of the Goldstone theorem and our formula (5), there is an important physical difference. It is best illustrated by working out the example of the quantum

wire. There is, in fact, no spontaneous symmetry breaking in the one-dimensional transport problem. The axial symmetry group of this system is U(1). When one introduces an operator $a(x)$ solving the continuity equation, one decompactifies the axial group from U(1) to R . Indeed, under the action of the symmetry generator $Q_L - Q_R$ the field $a(x)$ is shifted by a constant, $[Q_L - Q_R, a(x)] = i/\pi$. The group U(1) has no representations of this type; a constant cannot be in the same multiplet as a nontrivial field. By introducing the unphysical field $a(x)$, we effectively replace U(1) by its covering group R . Of course, it is not surprising to find that the expectation value of a constant is nonvanishing. But this fact is not related to any physical symmetry breaking. Note, that this situation is special for Abelian symmetry groups.

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