Analytical Solutions of Layzer-Type Approach to Unstable Interfacial Fluid Mixing

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We extend the Layzer-type approach to unstable interfacial fluid mixing, applied up to now only to vacuum bubbles, to spikes and derive the analytical solutions of the model for the positions, velocities, accelerations, and curvatures at the tips of the bubble and spike over all times. The analytical predictions are in good agreement with the results from numerical simulations for both spikes and bubbles. We give the first analytical prediction for the asymptotic growth rate of a spike at the Richtmyer-Meshkov unstable interface. We predict that, in contrast to the asymptotic bubble growth rate, the asymptotic growth rate of a spike at the Richtmyer-Meshkov unstable interface is a constant and depends on the initial condition. [S0031-9007(98)07317-7]

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It is well known that a material interface driven by an external force pointing from the heavy fluid to the light fluid or by a shock wave is unstable. The former is known as Rayleigh-Taylor (RT) instability [1,2], and the latter is known as Richtmyer-Meshkov (RM) instability [3,4]. Both these two instabilities play an important role in the study of supernova, inertial confinement fusion. Recent progress on the study of RT and RM instabilities can be found in and traced from [5–14].

Spikes and bubbles are formed at the unstable material interface. A bubble (spike) is a portion of light (heavy) fluid penetrating into heavy (light) fluid. A Layzer-type approach studies the motion at the tip of the bubble in a system with an infinite density ratio (the motion of a vacuum bubble). It approximates the shape of the material interface near the tip of the bubble as a parabola and derives a set of ordinary differential equations which determines the position, velocity, and curvature at the tip of the bubble. This approach was first introduced by Layzer for a bubble in Rayleigh-Taylor instability [15]. Alon *et al.* have extended the method to vacuum bubbles in RM instability and have shown that the model gives correct asymptotic bubble growth rates for both RT and RM instabilities [10,11].

So far, the Layzer-type model has been applied to bubbles only [10–12,15]. The solutions for bubbles can be found in a recent paper by Mikaelian [12]. In the case of RT instability, only the solution for a special initial condition has been found (the initial curvature of the bubble was set to its asymptotic limit) [12]. In this Letter, we show, for the first time, that the Layzer-type potential flow model is applicable to spikes as well. Furthermore, we derived the analytical solutions of the model for both spikes and bubbles over all times for all initial conditions. The predictions of our analytical solutions are in good agreement with the results of the numerical solutions of the Euler equations for both RT and RM instabilities and for both spikes and bubbles.

In the case of Richtmyer-Meshkov instability, it has been shown [10] that the asymptotic bubble growth rate is given by $2/3kt$, where k is the wave number. In this Letter, we give the first theoretical prediction that the asymptotic growth rate of the spike at the Richtmyer-Meshkov unstable interface is given by $\nu^{\mathrm{sp}}(t \to \infty) = \nu_0[(6\xi_0 + 3)/(6\xi_0 + 1)]^{1/2}$. Here ν_0 is the initial velocity perturbation, and ξ_0 is related to the initial curvature at the tip of the spike. Therefore, in contrast to the asymptotic bubble growth rate, the asymptotic spike growth rate does depend on the initial conditions ν_0 and ξ_0.

Following [15] and [10], we consider impressible, inviscid, and irrotational fluids with infinite density ratio. The governing equations for this system are

$$\nabla^2 \phi(x,z,t) = 0\,,$$

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z} = 0, \quad \text{at } z = \eta\,;$$

$$-g\eta + \frac{\partial \phi}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = \text{const}\,, \quad \text{at } z = \eta\,.$$

Here $z = \eta(x,t)$ is the position of the interface at time t. ϕ is the velocity potential.

Under a Layzer-type approximation, one expresses the potential ϕ as $\phi(t,x,z) = a(t)\cos(kx)e^{-kz}$, and approximates the shape of the interface near a finger (either a spike or a bubble) as a parabola, $\eta(t,x) = z_0(t) + \xi(t)kx^2$. In [10–12,15], the finger is a bubble. Here the finger can be either a bubble or a spike. After substituting these expressions into the above governing equations and expanding the resulting equations through the order of x^2, we have the following ordinary differential equations:

$$\frac{dz_0}{dt} - ake^{-kz_0} = 0\,, \tag{1}$$

$$\frac{d\xi}{dt} + ak^2 e^{-kz_0}\left(3\xi + \frac{1}{2}\right) = 0\,, \tag{2}$$

$$g\xi + ke^{-kz_0}\left(\xi + \frac{1}{2}\right)\frac{da}{dt} - k^3\xi a^2 e^{-2kz_0} = 0\,. \tag{3}$$

We now solve Eqs. (1)–(3). Eliminating a from (1) and (2), we obtain

$$\xi = \left[\left(\xi_0 + \frac{1}{6}\right)e^{-3k(z_0 - z_0^0)} - \frac{1}{6}\right]. \tag{4}$$

Here $z_0^0 = z_0$ $(t = 0)$ and $\xi_0 = \xi$ $(t = 0)$. [For a sinusoidal perturbation at the interface with an initial amplitude a_0, $\xi_0 = -(1/2)a_0k$ for the bubble and $\xi_0 = (1/2)a_0k$ for the spike.] Eliminating a from (2) and (3), we have

$$\frac{d^2z_0}{dt^2} + \frac{k}{2\xi + 1}\left(\frac{dz_0}{dt}\right)^2 + \frac{2\xi}{2\xi + 1}\, g = 0\,. \tag{5}$$

Since $d^2z_0/dt^2 = (d\xi/dt)(d\nu/d\xi) = -(k/2)(3\xi + \frac{1}{2})(d\nu^2/d\xi)$ where $\nu = dz_0/dt$, Eq. (5) can be expressed as

$$-\frac{k^2}{4}(6\xi + 1)\frac{d\nu^2}{d\xi} + \frac{k^2}{2\xi + 1}\nu^2 + \frac{2\xi}{2\xi + 1}\, g = 0\,. \tag{6}$$

After solving Eq. (6), we obtain the velocity. The result is

$$\nu = \nu_0\left[\frac{9(2\xi_0 + 1)k\nu_0^2 - 6(6\xi_0 + 1)(z_0 - z_0^0)kg + 2(e^{3k(z_0 - z_0^0)} - 1)g}{3k\nu_0^2(6\xi_0 + 1 + 2e^{3k(z_0 - z_0^0)})}\right]^{1/2}, \tag{7}$$

where $\nu_0 = \nu(t = 0)$. From (4), (5), and (7), we determine the acceleration. The result is

$$\frac{d\nu}{dt} = -g + \frac{3e^{3k(z_0 - z_0^0)}}{6\xi_0 + 1 + 2e^{3k(z_0 - z_0^0)}}\left[g - \frac{9(2\xi_0 + 1)k\nu_0^2 - 6(6\xi_0 + 1)(z_0 - z_0^0)kg + 2(e^{3k(z_0 - z_0^0)} - 1)g}{3(6\xi_0 + 1 + 2e^{3k(z_0 - z_0^0)})}\right]. \tag{8}$$

Finally, we solve the relation between t and z_0 from (7), and the result is

$$t - t_0 = (k\nu_0)^{-1}\int_0^{k(z_0 - z_0^0)}\left[\frac{3k\nu_0^2(6\xi_0 + 1 + 2e^{3x'})}{9(2\xi_0 + 1)k\nu_0^2 - 6(6\xi_0 + 1)gx' + 2(e^{3x'} - 1)g}\right]^{1/2} dx'\,. \tag{9}$$

Therefore we obtained the analytical solutions for ξ, z_0, ν, and $d\nu/dt$ over all time. They are given by (4), (9), (7), and (8), respectively. To evaluate these analytical solutions, one chooses a value for z_0 ($z_0 > 0$ for bubble and $z_0 < 0$ for spike), then determines t from (9), ν from (7), $d\nu/dt$ from (8), and ξ from (4). We emphasize that these solutions are applicable to both spikes and bubbles. A bubble has initial conditions $\nu_0 > 0$, $z_0^0 \geq 0$, and $\xi_0 \leq 0$, while a spike has initial conditions $\nu_0 < 0$, $z_0^0 \leq 0$, and $\xi_0 \geq 0$.

Let us examine the asymptotic solution of the Layzer-type model. By taking the large time limit of (4), (7), (8), and (9), we have the following asymptotic limits:

$$\text{For bubble in RT instability } (g > 0)\text{:}\quad \nu_{\mathrm{RT}}^{\mathrm{bb}} \to \sqrt{\frac{g}{3k}} \quad\text{and}\quad \xi_{\mathrm{RT}}^{\mathrm{bb}} \to -\frac{1}{6}\,; \tag{10}$$

$$\text{For spike in RT instability } (g > 0)\text{:}\quad \frac{d\nu_{\mathrm{RT}}^{\mathrm{sp}}}{dt} \to -g \quad\text{and}\quad \xi_{\mathrm{RT}}^{\mathrm{sp}} \to \infty\,; \tag{11}$$

$$\text{For bubble in RM instability } (g = 0)\text{:}\quad \nu_{\mathrm{RM}}^{\mathrm{bb}} \to \frac{2}{3kt} \quad\text{and}\quad \xi_{\mathrm{RM}}^{\mathrm{bb}} \to -\frac{1}{6}\,; \tag{12}$$

$$\text{For spike in RM instability } (g = 0)\text{:}\quad \nu_{\mathrm{RM}}^{\mathrm{sp}} \to \nu_0\left(\frac{6\xi_0 + 3}{6\xi_0 + 1}\right)^{1/2} \quad\text{and}\quad \xi_{\mathrm{RM}}^{\mathrm{sp}} \to \infty\,; \tag{13}$$

Therefore at late time the shape of the spike becomes a very long filament as it has been observed in full numerical simulations. For RT unstable systems, the spike reaches an asymptotic constant acceleration, and the bubble reaches an asymptotic growth rate. For RM unstable systems, the bubble growth rate decays to zero asymptotically, while the spike reaches a constant growth rate asymptotically. To the author's knowledge, Eq. (13) is the first theoretical prediction for the asymptotic growth rate of a spike at an RM unstable interface. It is interesting to note that the asymptotic growth rate of the spike in an RM unstable system depends on the initial conditions ν_0 and ξ_0, while the asymptotic growth rate of the bubble in an RM unstable system is independent of initial conditions (as long as $\nu_0 \neq 0$). The asymptotic acceleration of the spike and the asymptotic growth rate of the bubble in an RT unstable system do not depend on the initial conditions either. At finite time scales, the solutions for the bubble and spike depend on initial conditions.

There are two special cases in which the function $F()$ can be expressed in terms of elementary functions. The first case is $g = 0$ (RM instability). In this case (9) can be expressed as

$$t - t_0 = c_1 \frac{2}{3k}[(c_2 + e^{3kz_0})^{1/2} - (c_2 + e^{3kz_0^0})^{1/2} - \sqrt{-c_2}[\cos^{-1}(\sqrt{-c_2}\, e^{-(3k/2)z_0}) - \cos^{-1}(\sqrt{-c_2}\, e^{-(3k/2)z_0^0})] \quad \text{for } \xi_0 < -1/6; \tag{14}$$

$$t - t_0 = \frac{2}{3k} c_1 (e^{(3/2)kz_0} - e^{(3/2)kz_0^0}) \quad \text{for } \xi_0 = -1/6; \tag{15}$$

$$t - t_0 = c_1 \frac{2}{3k}\left[(c_2 + e^{3kz_0})^{1/2} - (c_2 + e^{3kz_0^0})^{1/2} - \sqrt{c_2}\,\ln\left(\frac{\sqrt{c_2 + e^{3kz_0}} + \sqrt{c_2}}{\sqrt{c_2 + e^{3kz_0^0}} + \sqrt{c_2}}\right) + \frac{3k}{2}\sqrt{c_2}\,(z_0 - z_0^0)\right] \quad \text{for } \xi_0 > -1/6. \tag{16}$$

Here $c_1 = (e^{-(3k/2)z_0^0}/\nu_0)(3\xi_0 + 3/2)^{-1/2}$ and $c_2 = (3\xi_0 + 1/2)e^{3kz_0^0}$ are constants.

The second case is $\xi_0 = -1/6$. In this case, ν, z_0, and $d\nu/dt$ can be expressed in terms of t:

$$z = z_0 + \frac{2}{3k}\ln\left[\frac{1}{c_4} f_1\left(\left|\frac{3kg}{4}\right|^{1/2}(t - t_0)\right) - f_2(c_4)\right], \tag{17}$$

$$\nu = \left\{\left[\frac{g}{3k} + \left(\nu_0^2 - \frac{g}{3k}\right) \times \left[\frac{1}{c_4} f_1\left(\left|\frac{3kg}{4}\right|^{1/2}(t - t_0)\right) - f_2(c_4)\right]^{-2}\right\}^{1/2}, \tag{18}$$

$$\frac{d\nu}{dt} = -\frac{1}{2}(3k\nu_0^2 - g) \times \left[\frac{1}{c_4} f_1\left(\left|\frac{3kg}{4}\right|^{1/2}(t - t_0)\right) - f_2(c_4)\right]^{-2}, \tag{19}$$

$$\xi = \xi_0 = -\frac{1}{6}. \tag{20}$$

Here $c_4 = |g/(g - 3k\nu_0)|^{1/2}$. $f_1(\cdot) = \cos(\cdot)$ and $f_2(\cdot) = \arccos(\cdot)$ when $g < 0$, which is the stable case. $f_1(\cdot) = \sinh(\cdot)$ and $f_2(\cdot) = \operatorname{arcsinh}(\cdot)$ when $3k\nu_0^2 \geq g \geq 0$. $f_1(\cdot) = \cosh(\cdot)$ and $f_2(\cdot) = \operatorname{arccosh}(\cdot)$ when $g \geq 3k\nu_0^2 \geq 0$. Since here $\xi = -1/6 < 0$, the expressions given by (17)–(20) are valid only for bubbles. The solutions for these two special cases can also be found in [12].

To verify the validity of the analytical solutions of our Layzer-type model, we compare our analytical predictions with the results from full numerical simulations. There are four full nonlinear numerical simulations available for the case of infinite density ratio: two for RT instability and two for RM instability. These full numerical simulations are based on the method of conformal mapping [16] and the finite difference method [11]. In these studies, the initial conditions of the spike and the initial conditions of the bubble have the same magnitude but opposite signs. Therefore only the initial conditions of the bubble will

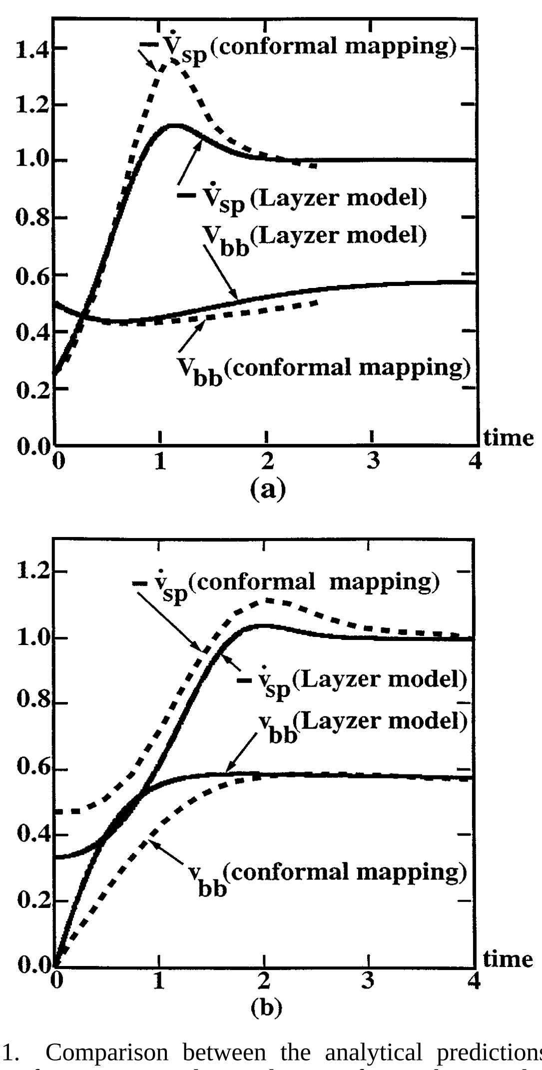

FIG. 1. Comparison between the analytical predictions and results from numerical simulations for spike acceleration and bubble velocity in Rayleigh-Taylor unstable systems. The numerical results are taken from [16]. In (a) the physical parameters are $g = k = 1$. The initial conditions are $\nu_0 = 1/2\ (-1/2)$, $z_0^0 = 0$, and $\xi_0 = 0$ for the bubble (spike). In (b) the physical parameters are $g = k = 1$. The initial conditions are $\nu_0 = 0$, $z_0^0 = 1/2\ (-1/2)$, and $\xi_0 = -1/4\ (1/4)$ for the bubble (spike).

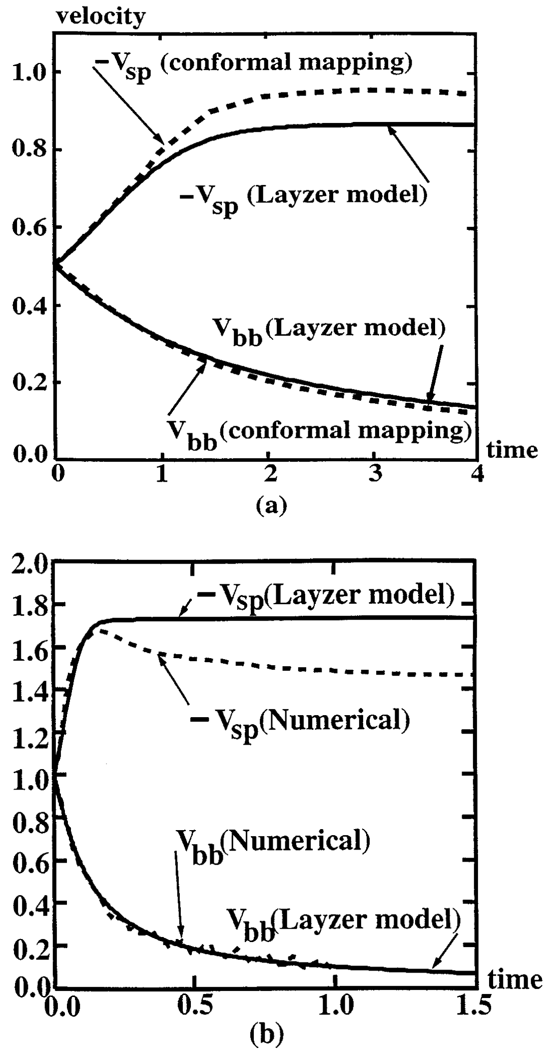

FIG. 2. Comparison between the analytical predictions and results from numerical simulations for spike velocity and bubble velocity in Richtmyer-Meshkov unstable systems. The numerical results are taken from [10,11,16]. In (a) the physical parameters are $g = 0$, $k = 1$. The initial conditions are $\nu_0 = 1/2$ $(-1/2)$, $z_0^0 = 0$, and $\xi_0 = 0$ for the bubble (spike). In (b) the physical parameters are $g = 0$ and $k = 2\pi$ cm^{-1}. The initial conditions are $\nu_0 = 1.0$ (-1.0) cm/ms, $z_0^0 = 0$, and $\xi_0 = 0$ for the bubble (spike).

be given here. In Fig. 1, we show the comparison for RT unstable systems. In Fig. 1(a), the physical parameters for the bubble are $g = k = 1$, $\nu_0 = 1/2$, $z_0 = 0$, and $\xi_0 = 0$. In Fig. 1(b), the physical parameters for the bubble are $g = k = 1$, $\nu_0 = 0$, $z_0 = 1/2$, and $\xi_0 = -1/4$. In Fig. 2, we show the comparison for RM unstable systems. In Fig. 2(a), the physical parameters for the bubble are $g = 0$, $k = 1$, $\nu_0 = 1/2$, $z_0 = 0$, and $\xi_0 = 0$. In Fig. 2(b), the physical parameters for the bubble are $g = 0$, $k = 2\pi$ cm^{-1}, $\nu_0 = 1.0$ cm/ms, $z_0 = 0$, and $\xi_0 = 0$. The numerical solutions shown in Figs. 1(a), 1(b), and 2(a) were obtained from conformal mapping [16]. The numerical solutions shown in Fig. 2(b) were obtained from finite difference method [10] (for the spike) and [11] (for the bubble after an appropriate rescaling of time and velocity). From Figs. 1 and 2, one can see that the agreement between the predictions from the analytical solutions and results from the numerical computations is reasonably good for both bubbles and spikes. The cases of $\xi_0 = 0$ shown in Figs. 2(a) and 2(b) are the only data sets available for infinite density ratio systems with zero gravity. Since (14) is nonsingular at $\xi_0 = 0$, Eq. (14) should give a good prediction at least for small values of ξ_0. However, because of the lack of the numerical simulations for such systems, the validity of prediction for asymptotic spike growth rate in RM instability over the range of nonzero ξ_0 cannot be established in this study. Further numerical simulations and validation studies for such systems are called for.

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