Stable Steady Flows in Rayleigh-Taylor Instability

S. I. Abarzhi*
Department of Mathematics, CB 3250, #411, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27599
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Steady flows generated by the Rayleigh-Taylor instability (RTI) are considered for incompressible inviscid fluid. There is a family of steady solutions, and for the first time the problem of the solutions' stability is studied theoretically for 2D and 3D flows. The region of stable solutions is found to be very narrow and bounded by Hopf' bifurcations. The influence of flow symmetry and discontinuity of dimensional crossover in RTI are shown; agreement with existing experimental and numerical data is good. Bubbles dynamics is discussed. [S0031-9007(98)06179-1]

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The Rayleigh-Taylor instability (RTI) is the instability of "a water layer on a ceiling." Internal fusion, astrophysics, plasma, and lasers are only a few of the instability applications [1]. We consider the instability for deep, inviscid, tensionless, and incompressible media differing greatly in densities, for example, water-air [1–3]. In accordance with linear theory [4], small perturbations of the media interface grow exponentially. Subsequently, the interface develops into a spatially periodic shape with rising bubbles of air and falling jets of water. The fluid motion at this stage is steady [1–3]. At later times, subharmonic modulations and the Kelvin-Helmholtz instability result in the flow inverse cascade and turbulent mixing [5].

Taylor was the first to observe the steady motion experimentally for a flow in a cylindrical tube [2]; in later experiments periodic steady flows were also found [3]. Although theories [6–8] and numerical simulations [9] gave reasonably agreeing results, full understanding of the phenomenon is not yet reached. One of the most interesting theoretical problems generated by the Rayleigh-Taylor instability is a uniqueness of steady solution. In 1957, Garabedian [7] predicted the existence of a one-parameter family of steady solutions for 2D flow, but only a few years ago the hypothesis was performed quantitatively [10]. The recent research dramatized the problem of RT steady motion for three-dimensional flow. It turned out that the dimension of steady solutions multitude (value n of n-parameters family of solutions) cannot be a "universal" instability property and is determined by the flow spatial symmetry [11]. The Froud number of steady flow depends on parameter or, as required by symmetry, on parameters. The curvature radius (radii) at the bubble top is that physical parameter(s) describing steady solutions. "Solitary jets" and "narrow bubbles" are the critical regions of the steady solutions family. Observed experimental and numerical data are located in the region of narrow bubbles. Nevertheless, no criterion separates a unique significant solution among the family. This work is devoted to linear stability analysis of steady "bubbles-jets" structures generated by the Rayleigh-Taylor instability. Having appeared originally in 1957, this problem never has been seriously studied.

Under some general requirements [12], the spatially periodic Rayleigh-Taylor instability can be considered on the basis of symmetry theory [13]. These requirements account for the weak modes coupling and the flow stability with respect to spatial noise at least at finite time. In a given experimental region, these conditions are met for a small amplitude initial perturbation with that symmorphic symmetry group posing a central point such as group $p6mm$ ($p4mm$, $p2mm$, etc.) of 3D flow or group $pm11$ in the degenerate case of 2D plane flow. One can state that for initial perturbation invariant with respect to one of these groups the steady motion is observed, and the flow spatial symmetry and translations are conserved up to the steady motion stage [12].

In moving with velocity $v(t)$ framework, the inviscid, tensionless, incompressible fluid motion with potential $\Phi(x, y, z, t)$ is described by the Laplace equation with the boundary conditions at the infinity and at the free fluid surface $z - z^*(x, y, t) = 0$:

$$\Delta\Phi = 0, \qquad \nabla\Phi|_{z=+\infty} = -v(t),$$
$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + \left(g + \frac{\partial v}{\partial t}\right)z|_{z=z^*} = 0, \qquad (1)$$
$$\frac{\partial z^*}{\partial t} + \nabla z\nabla\Phi|_{z=z^*} = 0.$$

Let the fluid motion be periodic in plane (x, y). With $v(t)$ taken as a velocity at the tops of rising bubbles in laboratory framework, the tops of bubbles in the moving framework (1) are stagnation points. We consider the advanced stage of RTI at times $t \gg \sqrt{\lambda/g}$ (λ is the flow wavelength), when $\partial\Phi/\partial t \to 0$ and $\partial z^*/\partial t \to 0$, and the fluid motion becomes steady.

The periodic potential (1) of 3D flow with hexagonal symmetry group $p6mm$ has the form

$$\Phi(x,y,z,t) = \sum_{m=0}^{\infty} \Phi_m(t) \times \left(3z + \frac{\exp(-mkz)}{mk} \sum_{i=1}^{3} \cos(m\boldsymbol{k}_i \boldsymbol{r})\right) + \text{cross terms}. \quad (2)$$

Here $\boldsymbol{k}_i$ are vectors in the inverse lattice with $\boldsymbol{k}_3 = \boldsymbol{k}_1 - \boldsymbol{k}_2$, $\boldsymbol{r} = (x,y)$. For $i,j = 1,2$, $\boldsymbol{k}_i \boldsymbol{a}_j = 2\pi\delta_{ij}$, where $\boldsymbol{a}_i$ are the translation vectors in the (x,y) plane with $|\boldsymbol{a}_1| = |\boldsymbol{a}_2| = \lambda$, and $\widehat{\boldsymbol{a}_1\boldsymbol{a}_2} = 2\pi/3$; $|\boldsymbol{k}_i| = k = 4\pi/(\lambda\sqrt{3})$. $\{\Phi\}$ is the Fourier amplitudes matrix. In the unit cell, we asymptotically expand the free boundary in the bubble top neighborhood:

$$z^*(\boldsymbol{r},t) = \sum_n \zeta_n(t) \sum_{i=1}^{3} (\boldsymbol{k}_i \boldsymbol{r}/k)^{2n} + \text{cross terms}. \quad (3)$$

Let us formulate the problem (1) in terms of time-dependent correlation functions or moments $\boldsymbol{M}(t)$. These functions $\boldsymbol{M}(t)$ are generated by Fourier amplitudes with "diagonal" moments $M_n(t) = \sum_m \Phi_m(t)(km)^n$ + cross terms, and the velocity $v(t) = -3M_0(t)$. Some of the moments $\boldsymbol{M}(t)$ are linear dependent because of the symmetry. Near the bubble top, the free-surface conditions of the problem (1) can be evolved by successive approximations into the form $\sum_{i,j} x^{2i}y^{2j} D_{ij}[\partial\boldsymbol{M}/\partial t, \boldsymbol{M}(t), \zeta(t)] = 0$, and $\sum_{i,j} x^{2i}y^{2j} K_{ij}[\partial\zeta/\partial t, \boldsymbol{M}(t), \zeta(t)] = 0$, where $i + j = N = 1,2,\ldots$, respectively. At any finite N, these equations describe a motion in a functional space of moments $\boldsymbol{M}(t)$ and surface variables $\zeta(t)$. One finds

$$\frac{3\boldsymbol{r}^2}{2}\left(-\frac{3\partial M_0(t)}{\partial t}\zeta_1(t) - \frac{\partial M_1(t)}{2\partial t} + g\zeta_1(t) + \frac{3M_1^2(t)}{4}\right) + \frac{9\boldsymbol{r}^4}{8}\left(-\frac{3\partial M_0(t)}{\partial t}\zeta_2(t) + \cdots\right) + \cdots = 0,$$
$$\frac{3\boldsymbol{r}^2}{2}\left(\frac{\partial\zeta_1(t)}{\partial t} - 6\zeta_1(t)M_1(t) - \frac{M_2(t)}{2}\right) + \frac{9\boldsymbol{r}^4}{8}\left(\frac{\partial\zeta_2(t)}{\partial t} + \cdots\right) + \cdots = 0. \quad (4)$$

Small deviations from steady solutions (1) have the form

$$\Phi_m(t) = \Phi_m^{(s)} + \Phi_m'(t) \quad \text{and} \quad \zeta_n(t) = \zeta_n^{(s)} + \zeta_n'(t). \quad (5)$$

The flow (1) is smooth, and in the stability region there are the relations

$$|\Phi_{m+1}^{(s)}| \ll |\Phi_m^{(s)}|, \quad |\zeta_{n+1}^{(s)}| \ll |\zeta_n^{(s)}|$$
$$\text{with } |\Phi_{m+1}'(t)| \ll |\Phi_m'(t)|, \quad |\zeta_{n+1}'(t)| \ll |\zeta_n'(t)|. \quad (6)$$

The flow properties (6) allow one to establish some additional relationships between correlators $\boldsymbol{M}(t)$ and their time derivatives at any finite N. At $N = 1$, $M_1(t) = kM_0(t) + k\Phi_2(t)$, and $M_2(t) = k^2M_0(t) + 3k^2\Phi_2(t)$. With neglected high-order terms in time derivatives of $\boldsymbol{M}(t)$, one finds

$$\frac{\partial M_0(t)}{\partial t}\left(3\zeta_1(t) + \frac{k}{2}\right) = g\zeta_1(t) + \frac{3}{4}k^2[M_0(t) + \Phi_2^{(s)}]^2,$$
$$\frac{\partial\zeta_1(t)}{\partial t} = 6k\zeta_1(t)[M_0(t) + \Phi_2^{(s)}] + \frac{k^2[M_0(t) + 3\Phi_2^{(s)}]}{2}. \quad (7)$$

We will consider "hexagonal" steady solutions $\{\Phi^{(s)}, \zeta^{(s)}\}$ in detail and convergence treatment elsewhere [12]. Briefly, the curvature radius R at the bubble top is the parameter of the steady solutions family (1)–(3). In the physical region $R_{\mathrm{cr}} < R < \infty$, the velocity dependence on the parameter is nonmonotone. At $kR_{\mathrm{cr}} = (2.716 \pm 0.316)$ with $v_{\mathrm{cr}} = (0.955 \pm 0.035)\sqrt{g/k}$, the conditions (6) are broken. At $kR_{\max} = 4$, maximum velocity $\sqrt{g/k}$ reproduces at $N = 1$ renormalized Layzer's result [6], while with a rise in N this value slightly increases; $v_{\max} \approx 1.05\sqrt{g/k}$ [12]. At $R_{\mathrm{cr}} \to \infty$ (solitary jets) and at $R_{\mathrm{cr}} \le R \le R_{\max}$ (narrow bubbles), the convergence of approximated steady solutions is a good, exponential one, while the region of medium curvature radii, $4.9 < kR < 9.2$, is poorly approximated [12].

At $N = 1$, the family of steady solutions (1)–(3), (7) has the form $\zeta_1^{(s)} = -1/3R$, $v^{(s)} = \sqrt{g/k}\,(3kR - 4)/(kR)^{3/2}$. In this way, the eigenvalues (Lyapunov' exponents) of Eqs. (7) are easily deduced:

$$w_1^{(1)} = \sqrt{gk}\,\frac{A_h + B_h}{C_h} \quad \text{and} \quad w_2^{(1)} = \sqrt{gk}\,\frac{A_h - B_h}{C_h}. \quad (8)$$

Here $A_h = -3kR + 4$, $B_h = \sqrt{(kR)^3 - 5(kR)^2 + 16}$, and $C_h = (kR - 2)\sqrt{kR}$. Expressions (8) lead one to a clear physical result: The steady solutions with either very large curvature radii, $12 < kR < \infty$, or very small ones, $0 < kR < 2$, appeared unstable. Thus, in the first rough approximation, the region of stable physical solutions lies in the interval $kR_{\mathrm{cr}} \le kR \le 12$. At $N = 2$, the stability region is narrowed to the vicinity of the point $R_{\max} \approx 4/k$, associated with maximum velocity. Only those solutions with parameter values $3.356 \le kR \le 5.184$ are stable, and distinctive points $R_c^{(1)} = 3.356/k$ and $R_c^{(2)} \le 5.184/k$ are points of Hopf' bifurcations or "limit circles" (Fig. 1). The right bifurcation $R_c^{(2)}$ takes out a poorly approximated region of medium curvature radii from the stability region. The left limit cycle, $R_c^{(1)} = 0.925(\lambda/2)$ with $v_c^{(1)} = 1.013\sqrt{g/k} = 0.532\sqrt{g(\lambda/2)}$, is located in the narrow bubbles region. The dependence of velocity on

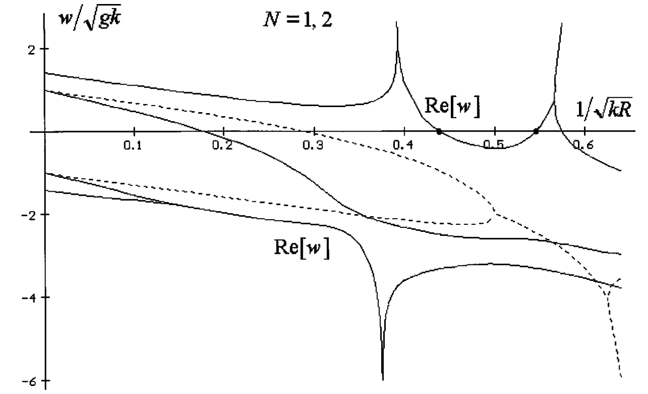

FIG. 1. Stability of steady solution. Dependencies of eigenvalues w on parameter R. 3D flow with hexagonal symmetry $p6mm$. Dashed lines: $N = 1$; solid lines: $N = 2$. At $N = 2$, the stable steady solutions are intervening between two Hopf bifurcations (black points). $R_c^{(1)} = 3.356/k$, $R_c^{(2)} = 5.184/k$, and $R_l \to 4/k$.

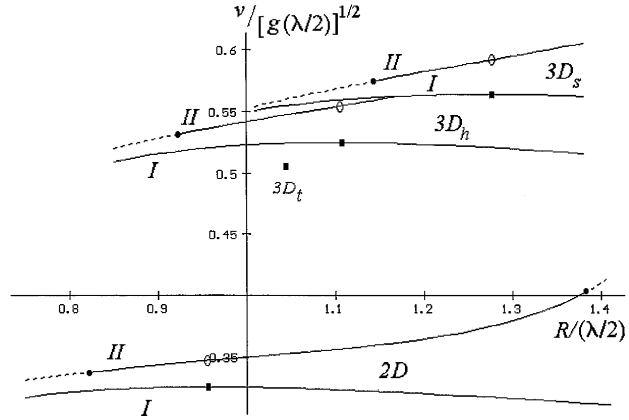

FIG. 2. Influence of flow symmetry on steady solutions in RTI. Hexagonal $3D_h$, square $3D_s$, and plane 2D flows. Roman numerals denote approximations. Solid lines are stable solutions; dashed lines are unstable solutions. Black points are Hopf' bifurcations at $N = 2$. Black squares are renormalized Layzer's solutions, $v = \sqrt{g/k}$, $R = 4/k$ in 3D ($3D_t$—tubular flow), and $v = \sqrt{g/3k}$, $R = 3/k$ in 2D. The circles are limiting renormalized solutions: $v_{l,3D} \approx 1.05\sqrt{g/k}$, $R_{l,3D} = 4/k$ and $v_{l,2D} \approx 1.06\sqrt{g/k}$, $R_{l,2D} = 3/k$.

the parameter in this region is weak, and the "bifurcated" value of velocity $v_c^{(1)}$ is close numerically to maximal one.

We apply the outlined method to study the stability of 2D and 3D steady flows with various symmetries. Obtained results share a number of common properties. If symmetry poses a one-parameter family of solutions, like "square" 3D ($p4mm$) or "plane" 2D ($pm11$) flows, then narrowing of the stability region with a rise in approximations as well as "eigenmode" interactions will be very similar to those of the hexagonal flow. For 3D square flow $p4mm$, $k = 2\pi/\lambda$, at $N = 2$, limit cycles $R_c^{(1)} = 3.588/k = 1.142(\lambda/2)$ $[v_c^{(1)} = 1.018\sqrt{g/k} = 0.575\sqrt{g(\lambda/2)}]$, and $R_c^{(2)} = 5.050/k$ bound the interval with inner point $R_{\max} = 4/k$ related to velocity $v_{\max} \approx 1.05\sqrt{g/k}$. Note that in dimensionless units $\sqrt{g/k}$ and $1/k$, square and hexagonal results are weakly differing: Bubbles are near-axisymmetric for both symmetries, $z^* \sim -(x^2 + y^2)/2R$. Nevertheless, the value of wave vector k depends significantly on a lattice (tube), and curvature radii $[R_c^{(1)}]_{p4mm}$:$[R_c^{(1)}]_{p6mm} \approx 1.23$ in λ units. Moreover, comparison with Taylor experiment, $v_T = 0.49\sqrt{g(\lambda/2)}$, shows the influence of 3D flow symmetry (Fig. 2). Obtained fluid behavior is a remarkable one. When a tubular flow is destroyed by a lattice, bubbles remain near-axisymmetric and become wider, jets tend to be more singular, and steady velocity increases. These changes in curvature radius and velocity are sufficiently large to be observable. Among all kinds of lattices, the hexagonal is the closest to tubular flow.

Brightly, a tendency to conserve a near-axisymmetric bubble manifests for elongated lattice, symmetry $p2mm$, and for 3D-2D transition [12]. In this case, the steady solutions family is described by two independent parameters or curvature radii R_x, R_y [11]. At finite values λ_x/λ_y, steady solutions with either $R_x \to \infty$ or $R_y \to \infty$ (plane in x or y direction flows, respectively) are unstable. When $\lambda_x/\lambda_y \sim 1$, stable solutions are located in an "axisymmetric" region $R_y \sim R_x(\lambda_y/\lambda_x)^2$, while in the limiting case of 3D-2D dimensional crossover, $(\lambda_x/\lambda_y) \to 0$, no stable steady solutions exist. The eigenmode associated with 3D-2D transitions equals zero at $(\lambda_x/\lambda_y) \equiv 0$ (2D flow) and is positive at $(\lambda_x/\lambda_y) \ll 1$ (3D flow) [12]. Thus, stable steady bubbles are allowed by "rectangular" symmetry only at finite values of aspect ratio λ_x/λ_y, and dimensional crossover for RT bubbles is discontinuous. This result is reasonable: The topological structure of 2D bubbles is different than of 3D bubbles. At $N = 1$, we can roughly evaluate the cutoff as $\lambda_x/\lambda_y \sim 0.2$–$0.5$ [12]. We believe that at $N \to \infty$ the cutoff λ_x/λ_y takes a value of order of 1.

3D experiments and numerical simulations in RTI are difficult to perform and rare. Existing experimental and numerical data support the above theoretical prediction [2,3,14,15]. For example, the quantitative disagreement between steady velocities of the square bubble and Taylor's experiment has been shown in 3D simulations by Li [14]. The values of the square bubble radius obtained numerically in [15], $R \approx (3.6 \pm 0.3)/k$, are close to our stable theoretical solutions. Decaying instability of elongated bubbles is another important fact observed but not appreciated in [14]. The elongation of the bubbles initially posed in [14] results very fast in the relation $R_y \gg R_x(\lambda_y/\lambda_x)^2$, $\lambda_y/\lambda_x > 1$, and, in accordance with our analysis, these bubbles become unstable in the advanced instability stages. It should be noted that the tendency to conserve a near-circular contour is the physical reason of the decreasing of the flow scale λ_y and the splitting of elongated bubbles in [14].

2D steady flows in RTI are studied numerically and theoretically in more detail [1,6–10], and our stability analysis is of crucial interest for 2D flow $pm11$. Similarly, 3D flows $p6mm$ and $p4mm$, for 2D flow $pm11$, at $N = 2$, the limit circles $R_c^{(1)} = 2.576/k$, and $R_c^{(2)} = 4.340/k$ bound the interval with an inner point related to Layzer's solution ($v_{\max} = \sqrt{g/3k}$ with $R_{\max} = 3/k$ at $N = 1$ [6,10,12]) (Fig. 2). The bifurcation $R_c^{(1)} = 2.576/k = 0.820(\lambda/2)$ has velocity $v_c^{(1)} = 0.598\sqrt{g/k} = 0.337\sqrt{g(\lambda/2)}$ and Froud $\mathrm{Fr}_c^{(1)} = 0.238$. These values are very close to existing data [1,6–9].

In all of the above cases at higher orders of approximations, $N > 2$, numerical solutions are required. Nevertheless, it can be shown that, at finite N, the system (1) eigenmodes are complex functions of parameter(s) because of variables $\boldsymbol{M}(t)$ and $\zeta(t)$ "meshing." We expect that at $N \to \infty$ the stability region is narrowed, bifurcation points are brought together, and a "limit cycle" is a unique significant flow. In dimensionless variables, the curvature radius of the "limiting" bubbles approaches $R_l \to 4/k$ for 3D and $R_l \to 3/k$ for 2D flows and is associated with maximal velocity (Fig. 2). Although these values of R are separated by Layzer's and Taylor's considerations [2,6,12], our solutions, renormalized in the appropriate way, give a "unique" bubble velocity that slightly exceeds prediction [6]: $v_{l,3\mathrm{D}} \to 1.05\sqrt{g/k}$ and $v_{l,2\mathrm{D}} \to 1.06\sqrt{g/3k}$ (Fig. 2). It should be noted that the 2D limiting velocity $v_{l,2\mathrm{D}}$ coincides with Garabedian's prediction $\mathrm{Fr}_{G,2\mathrm{D}} \approx 0.24$ [7]. In this way, the stability analysis eliminates the discrepancies between previous theoretical approaches [6,7,9,10].

The above local stability analysis provides answers to the problem of uniqueness of steady solutions in the Rayleigh-Taylor instability. Spatial symmetry of 2D or 3D flow poses a family of steady solutions for tensionless inviscid fluid. Most of these solutions are unstable. There is the region of stable solutions with values of curvature radii at the bubble top running over a narrow interval. 3D steady bubbles in RTI are quite sensitive to symmetry breaking; they tend to conserve a near-circular contour and cannot be transformed into 2D bubbles continuously.

Our theoretical analysis clearly demonstrates that in real experiments on RT instability NO bubbles are 2D. A large-scale noise immediately converts the "approximately" 2D bubbles system into a 3D one followed by 3D bubbles splitting. These results, in fact, shed new light on 3D and 2D experiments on turbulent mixing by Read [3]. An equivalence of the directions normal to gravity $\boldsymbol{g}$ and a near-circular contour of the bubbles are the basic properties of the Rayleigh-Taylor instability. These properties must determine the instability dynamics from initial perturbation to turbulence. They define an attractor in a functional space, while bubbles merging (resulting in inverse cascade) and bubbles splitting (resulting in direct cascade) are those processes that form dynamic trajectories of the system. A correct statistical model of turbulent mixing in the Rayleigh-Taylor instability cannot ignore both of these processes and competition between them.

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*Electronic address: snezha@math.unc.edu

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