Eigenvector Statistics in Non-Hermitian Random Matrix Ensembles

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We study statistical properties of the eigenvectors of non-Hermitian random matrices, concentrating on Ginibre's complex Gaussian ensemble, in which the real and imaginary parts of each element of an $N \times N$ matrix, J, are independent random variables. Calculating ensemble averages based on the quantity $\langle L_\alpha | L_\beta \rangle \langle R_\beta | R_\alpha \rangle$, where $\langle L_\alpha |$ and $| R_\beta \rangle$ are left and right eigenvectors of J, we show for large N that eigenvectors associated with a pair of eigenvalues are highly correlated if the two eigenvalues lie close in the complex plane. We examine consequences of these correlations that are likely to be important in physical applications. [S0031-9007(98)07357-8]

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An understanding of statistical properties of ensembles of random matrices has proved useful in many different areas of physics [1]. Because the first application of random matrix theory, and one still of great importance, was to represent the Hamiltonian of a nonintegrable quantum system, the early work of Wigner, Dyson, and others focused on ensembles of real symmetric or complex Hermitian matrices. Eigenvector distributions in these ensembles are of limited interest, being determined by the Haar measure on the group that leaves the ensemble invariant. Instead, the concern is mainly with eigenvalue correlations, about which a great deal is now known [1].

More recently, the spectral properties of random non-Hermitian operators have attracted attention in a variety of contexts, including: neural network dynamics [2]; the quantum mechanics of open systems [3]; the statistical mechanics of flux lines in superconductors with columnar disorder [4–8]; classical diffusion in random media [9]; and biological growth problems [10]. The corresponding ensembles of real asymmetric and general complex matrices were first studied by Ginibre [11], Girko [12], and Sommers and co-workers [2,13]. The eigenvalues in these ensembles are, of course, not restricted to the real axis, but rather distributed over an area in the complex plane. Their density and correlations have been investigated in considerable detail [2,11–14].

By contrast, *eigenvector* statistics in non-Hermitian random matrix ensembles have not, so far as we know, previously been examined. The existence of distinct sets of left and right eigenvectors means that invariance of the ensemble, under $\mathrm{O}(N)$ or $\mathrm{U}(N)$ transformations as appropriate, is a rather weak constraint on the joint eigenvector distribution: it generates no information on the relative orientations of the two sets of vectors. We show in this paper that there are, in fact, remarkable correlations between left and right eigenvectors. These correlations are likely to be important in physical applications of non-Hermitian random matrix ensembles. We illustrate their significance by discussing two consequences, involving extreme sensitivity of spectra to perturbations, and transients in time-evolution, which are recognized in other contexts as typical of non-normal operators [15–17].

We consider, following Ginibre [11], the Gaussian ensemble of general complex $N \times N$ matrices, J, having independent matrix elements, J_{kl}, distributed with probability

$$P(J)\, dJ \propto \exp(-N\, \mathrm{Tr}[JJ^\dagger]) \prod_{k,l=1}^{N} dJ'_{kl}\, dJ''_{kl}\,, \qquad (1)$$

where $J_{kl} = J'_{kl} + iJ''_{kl}$, with J'_{kl} and J''_{kl} real. Denoting ensemble averages by $\langle \cdots \rangle$ and complex conjugation with an overbar, the only nonzero cumulant of J is $\langle J_{kl} \overline{J}_{kl} \rangle = 1/N$.

The eigenvalues, λ_α, of J are distributed in the complex plane with, in the limit $N \to \infty$, constant density inside a disc of unit radius, centered on the origin. They are nondegenerate with probability one, and in this case the left and right eigenvectors, $\langle L_\alpha |$ and $| R_\alpha \rangle$, which satisfy

$$\begin{aligned} J | R_\alpha \rangle &= \lambda_\alpha | R_\alpha \rangle\,, \\ \langle L_\alpha | J &= \langle L_\alpha | \lambda_\alpha \end{aligned} \qquad (2)$$

form two complete, biorthogonal sets, and can be normalized so that

$$\langle L_\alpha | R_\beta \rangle = \delta_{\alpha\beta}\,. \qquad (3)$$

We indicate Hermitian conjugates of vectors in the usual way, so that, for example, $| L_\alpha \rangle$ satisfies $J^\dagger | L_\alpha \rangle = \overline{\lambda}_\alpha | L_\alpha \rangle$.

We investigate eigenvector correlations mainly by calculating ensemble averages of combinations of scalar products. Noting that Eqs. (2) and (3) are invariant under a scale transformation $| R_\alpha \rangle \to \zeta_\alpha | R_\alpha \rangle$ and $\langle L_\beta | \to \langle L_\beta | \zeta_\beta^{-1}$, one recognizes that only those combinations invariant under this transformation should be considered. The simplest such combination, involving two eigenvectors, is fixed by Eq. (3); the simplest nontrivial quantity is thus the matrix of overlaps

$$O_{\alpha\beta} = \langle L_\alpha | L_\beta \rangle \langle R_\beta | R_\alpha \rangle \qquad (4)$$

and we shall focus on this throughout the paper. It is convenient to define local averages of diagonal and off-diagonal elements of the overlap matrix,

$$O(z) = \left\langle \frac{1}{N} \sum_{\alpha} O_{\alpha\alpha} \delta(z - \lambda_\alpha) \right\rangle, \quad (5)$$

$$O(z_1, z_2) = \left\langle \frac{1}{N} \sum_{\alpha \neq \beta} O_{\alpha\beta} \delta(z_1 - \lambda_\alpha) \delta(z_2 - \lambda_\beta) \right\rangle. \quad (6)$$

Correspondingly, the density of states is defined as $d(z) = \langle N^{-1} \sum_\alpha \delta(z - \lambda_\alpha) \rangle$. In the limit $N \to \infty$, $d(z) = \pi^{-1}$ for $|z| < 1$ and $d(z) = 0$, otherwise [11].

We have been able to obtain exact expressions for $O(z_1)$ and $O(z_1, z_2)$. For $N \gg 1$, $|z_1 - z_2| \neq 0$ and $|z_1|, |z_2| < 1$ these simplify to

$$O(z_1) = \frac{N}{\pi} (1 - |z_1|^2), \quad (7)$$

$$O(z_1, z_2) = -\frac{1}{\pi^2} \frac{1 - z_1 \bar{z}_2}{|z_1 - z_2|^4}. \quad (8)$$

For $|z_1|, |z_2| \geq 1$, both densities vanish as $N \to \infty$. To display the form of $O(z_1, z_2)$ as $|z_1 - z_2| \to 0$, it is necessary to express $|z_1 - z_2|$ in units of the separation between adjacent eigenvalues, introducing $z_+ = (z_1 + z_2)/2$ and $\omega = \sqrt{N}(z_1 - z_2)$. We obtain, for $|z_+| < 1$, $\omega \ll \sqrt{N}$, and $N \gg 1$

$$O(z_1, z_2) = -N^2 \frac{1 - |z_+|^2}{\pi^2 |\omega|^4} (1 - (1 + |\omega|^2) e^{-|\omega|^2}). \quad (9)$$

Equations (7)–(9) constitute our main results. Before outlining our derivation, we discuss their significance.

First, we stress the dramatic difference between the behavior of $O_{\alpha\beta}$ in this general complex ensemble and its behavior in the case of Hermitian matrices, for which $O_{\alpha\beta} = \delta_{\alpha\beta}$. The fact that, by contrast, $O_{\alpha\alpha} \sim N$ in the non-Hermitian ensemble can be understood as the behavior which results if $\langle L_\alpha|$ and $|R_\alpha\rangle$ are independent random vectors, subject to the normalization of Eq. (3). Moreover, large values for the diagonal elements of the matrix $O_{\alpha\beta}$ must be accompanied by some large (or many small) off-diagonal elements, since the two are linked by a sum rule that follows from completeness,

$$\sum_\alpha O_{\alpha\beta} = 1. \quad (10)$$

Indeed, Eq. (6) implies

$$O_{\alpha\beta} \sim O(z_1, z_2) \Bigg/ \left\langle \frac{1}{N} \sum_{\mu \neq \nu} \delta(z_1 - \lambda_\mu) \delta(z_2 - \lambda_\nu) \right\rangle, \quad (11)$$

and hence, from Eq. (9), $O_{\alpha\beta} \sim -N$ if λ_α and λ_β are neighboring eigenvalues in the complex plane, so that (typically) $\omega \sim 1$.

An immediate consequence of large values of $O_{\alpha\alpha}$ is that the spectrum of J for a given realization has extreme sensitivity to perturbations. Consider for definiteness $J = \cos(\theta) J_1 + \sin(\theta) J_2$, where θ is real and J_1 and J_2 are both drawn independently from the ensemble of Eq. (1), so that J moves through the ensemble as θ varies. Then

$$|\partial \lambda_\alpha / \partial \theta|^2 = |\langle L_\alpha | \partial J / \partial \theta | R_\alpha \rangle|^2. \quad (12)$$

Performing the ensemble average, and using Eq. (7) one obtains for the mean square eigenvalue velocity

$$\langle |\partial \lambda / \partial \theta|^2 \rangle = \frac{1}{\pi} (1 - |\lambda|^2). \quad (13)$$

This result should be contrasted in magnitude with the analogous one for Hermitian matrices [18]. Let $H = \cos(\theta) H_1 + \sin(\theta) H_2$, where H_1 and H_2 are complex Hermitian $N \times N$ matrices, drawn independently from the Gaussian unitary ensemble, in which the nonzero cumulants are $\langle H_{kl} H_{lk} \rangle \equiv \langle H_{kl} \overline{H}_{kl} \rangle = 1/N$. Let E be an eigenvalue of H. Then for $N \to \infty$, $-1 \leq E \leq 1$ and $\langle [\partial E / \partial \theta]^2 \rangle = 1/N$. Thus the eigenvalues of the $N \times N$ random non-Hermitian matrix are $\mathcal{O}(N)$ times more sensitive to perturbations than those of the Hermitian matrix. Such sensitivity is known to be a typical property of non-normal operators [17]. Despite the sensitivity of individual eigenvalues to perturbations, it is reasonable to expect some stability in the structure of the spectrum as a whole, since the perturbations considered merely take a random matrix from one realization to another. Such stability arises from the fact that, although the mean square velocity of Eq. (13) is large for eigenvalues within the unit disc, it vanishes as the boundary to the support of the spectrum is approached. Conversely, anticipating this stability, we have a rationalization of the fact that, from Eqs. (7), (9), and (11), the amplitudes of the $\mathcal{O}(N)$ contributions to $O_{\alpha\alpha}$ and $O_{\alpha\beta}$ vanish as $|z_1| \to 1$.

The large off-diagonal elements of $O_{\alpha\beta}$ are significant in situations in which J is the generator of evolution in real or imaginary time. Settings of this type represent one of the main physical applications of non-normal operators [2,4,9,10]. To be specific, consider a model problem in which

$$\frac{\partial}{\partial t} |u(t)\rangle = (J - 1) |u(t)\rangle \quad (14)$$

so that

$$|u(t)\rangle = \sum_\alpha |R_\alpha\rangle f_t(\lambda_\alpha) \langle L_\alpha | u(0) \rangle, \quad (15)$$

with $f_t(\lambda) = \exp([\lambda - 1]t)$, where we use $(J - 1)$ rather than J in Eq. (14) for convenience, to suppress exponential growth. Ensemble averaging with $\langle u(0) | u(0) \rangle = 1$ leads to

$$\langle\langle u(t) | u(t) \rangle\rangle = \left\langle \frac{1}{N} \sum_{\alpha\beta} O_{\alpha\beta} f_t(\lambda_\alpha) \overline{f}_t(\lambda_\beta) \right\rangle, \quad (16)$$

and for $t \gg 1$ and $N \to \infty$ we find [19]

$$\langle\langle u(t) | u(t) \rangle\rangle \sim (4\pi t)^{-1/2}. \quad (17)$$

This behavior should be compared with the much faster decay that would result from the same spectrum if the eigenvectors were orthogonal. In the same regime, the replacement $O_{\alpha\beta} \to \delta_{\alpha\beta}$ transforms Eq. (16) into

$$\left\langle \sum_{\alpha} |f_t(\lambda_\alpha)|^2 \right\rangle \sim (4\pi t^3)^{-1/2}. \tag{18}$$

Thus, eigenvector correlations may be as significant as eigenvalue distributions in determining evolution at intermediate times, a fact of established importance in hydrodynamic stability theory [15,16].

Finally, it is interesting to ask about, not only the average behavior of the overlap matrix, but also its fluctuations. In fact, $O_{\alpha\beta}$ is typically large if the matrix J has an eigenvalue which is almost degenerate with λ_α or λ_β, and as a result, the probability distribution of $O_{\alpha\beta}$ has a power-law tail extending to large $|O_{\alpha\beta}|$. To illustrate this, we consider $N = 2$, for which we can calculate exactly the probability distribution, $P(O_{\alpha\alpha})$, of a diagonal element of the overlap matrix. We find

$$P(O_{\alpha\alpha}) = 4\,\frac{\Theta(O_{\alpha\alpha} - 1)}{(2O_{\alpha\alpha} - 1)^3}, \tag{19}$$

where $\Theta(x) = 1$ for $x > 0$ and zero otherwise. This implies in particular that the second and higher moments of $O_{\alpha\alpha}$ diverge. We expect, from Eq. (20), below, similar behavior for $N > 2$ and (provided $N > 2$) for $O_{\alpha\beta}$ with $\alpha \neq \beta$.

In the remainder of this paper we sketch our calculations and show how the results summarized above can be generalized. Calculations for the ensemble of Eq. (1) can be done by extending the classical methods of Dyson and Ginibre, while more general problems are most conveniently treated via ensemble-averaged resolvents, using the techniques of Refs. [9,20,21].

A direct computation of averages of $O_{\alpha\beta}$ involves a $2N^2$-fold integration over the complex matrix elements J_{kl}. The integral is simplified considerably by changing variables as described in [22]. Reducing J by a unitary transformation, U, to upper triangular form, so that $T \equiv U^\dagger J U$ has $T_{kl} = 0$ for $k > l$, we use as $N(N + 1)$ coordinates, the real and imaginary parts of the nonzero elements of T_{kl}, and take the remaining coordinates from U itself. The required Jacobian is given by Mehta [22].

In this basis, the diagonal elements of T are the eigenvalues, $T_{kk} = \lambda_k$. The first two pairs of eigenvectors are $|R_1\rangle = (1, 0, \ldots, 0)^\dagger$, $\langle L_1| = (1, b_2, b_3, \ldots, b_N)$, $|R_2\rangle = (-\overline{b}_2, 1, 0, \ldots, 0)^\dagger$, and $\langle L_2| = (0, 1, d_3, \ldots, d_N)$, where the coefficients b_l and d_l are determined by the recursion relations

$$\begin{aligned} b_p &= \frac{1}{\lambda_1 - \lambda_p} \sum_{q=1}^{p-1} b_q T_{qp}, \\ d_p &= \frac{1}{\lambda_2 - \lambda_p} \sum_{q=1}^{p-1} d_q T_{qp}, \end{aligned} \tag{20}$$

with $b_1 = 1$, $d_1 = 0$, and $d_2 = 1$. Correspondingly, the overlaps are

$$O_{11} = \sum_{l=1}^{N} |b_l|^2, \tag{21}$$

$$O_{12} = -\overline{b}_2 \sum_{l=1}^{N} b_l \overline{d}_l. \tag{22}$$

Performing the integrals over U and T_{kl} with $k < l$, $O(z)$ and $O(z_1, z_2)$ can be expressed as averages with respect to the joint probability density of the eigenvalues

$$P(\lambda_1, \ldots, \lambda_N) \propto \exp\left(-N \sum_{k=1}^{N} |\lambda_k|^2\right) \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^2. \tag{23}$$

Defining $\langle \cdots \rangle_P$ as an average with (23), we find

$$O(z_1) = \left\langle \delta(z_1 - \lambda_1) \prod_{2 \le j \le N} \left(1 + \frac{1}{N|\lambda_1 - \lambda_j|^2}\right) \right\rangle_P \tag{24}$$

and

$$O(z_1, z_2) = -(N - 1) \left\langle \delta(z_1 - \lambda_1)\delta(z_2 - \lambda_2) \frac{1}{N|\lambda_1 - \lambda_2|^2} \prod_{3 \le j \le N} \left(1 + \frac{1}{N(\lambda_1 - \lambda_j)(\overline{\lambda}_2 - \overline{\lambda}_j)}\right) \right\rangle_P. \tag{25}$$

Performing the integrals over eigenvalues in Eqs. (24) and (25), we obtain explicit expressions in terms of $N \times N$ determinants.

For $z_1 = 0$, we are able to evaluate these determinants in closed form by recursion, and to simplify the result further for $N \gg 1$. For $z_1 \neq 0$ we are forced to take a less direct approach, which numerical tests show is a good approximation for finite N, and which we can prove is exact in the limit $N \to \infty$. We separate the contributions to each of the Eqs. (24) and (25) into two factors: one from the M eigenvalues closest to z_1, with $M \gg 1$, and another from the remaining eigenvalues. The first can be evaluated using our result for $z_1 = 0$, while the second can be calculated neglecting eigenvalue correlations, because its fluctuations vanish as $M \to \infty$. Their combination is independent of M, for M large, as it should be, and is as displayed in Eqs. (7)–(9).

An entirely different approach is necessary in order to treat other random matrix ensembles with ease, or to develop approximation schemes for spatially extended problems such as those of Refs. [2–4,9,10]. For these purposes we take as central objects the ensemble averages of products of the resolvents, $(z_1 - J)^{-1}$ and $(\overline{z}_2 - J^\dagger)^{-1}$. As a demonstration, we examine a matrix ensemble with the probability distribution

$$P(J)\,dJ \propto \exp\left(-\frac{N}{1-\tau^2}\,\mathrm{Tr}[JJ^\dagger - \tau\,\mathrm{Re}\,JJ]\right) \times \prod_{k,l=1}^{N} dJ'_{kl}\,dJ''_{kl}\,, \tag{26}$$

with τ real and $-1 \le \tau \le 1$. The nonzero cumulants are $\langle J_{kl}\overline{J}_{kl}\rangle = 1/N$, and $\langle J_{kl}\overline{J}_{lk}\rangle = \tau/N$. This distribution, introduced in [2], interpolates between the Gaussian unitary ensemble of Hermitian matrices, for $\tau = 1$, Ginibre's ensemble [Eq. (1)] for $\tau = 0$, and complex antisymmetric matrices for $\tau = -1$. In the limit $N \to \infty$, the eigenvalue density has the uniform value $d(z) = [\pi(1-\tau^2)]^{-1}$ within the ellipse defined by $[\mathrm{Re}\,z/(1-\tau)]^2 + [\mathrm{Im}\,z/(1+\tau)]^2 < 1$ and is zero elsewhere [2].

We treat the ensemble (26) using the techniques described in [9,20,21]. These generate an expansion for $O(z_1,z_2)$ in powers of $(z_1-z_2)/N$, and hence give $O(z_1,z_2)$ exactly in the limit $N\to\infty$, but supply information about $O(z)$ only indirectly, via the sum rule of Eq. (10). We start by considering the $2N \times 2N$ Hermitian matrix $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1$,

$$\boldsymbol{H}_0 = \begin{pmatrix} \eta & \\ & -\eta \end{pmatrix}, \qquad \boldsymbol{H}_1 = \begin{pmatrix} & z-J \\ \overline{z}-J^\dagger & \end{pmatrix}, \tag{27}$$

with real $\eta > 0$, and its inverse

$$\boldsymbol{G} = \boldsymbol{H}^{-1} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \tag{28}$$

The resolvents are obtained taking $\eta \to 0$: in this limit, $G_{21} = (z-J)^{-1}$ and $G_{12} = (\overline{z}-J^\dagger)^{-1}$. Expanding the Green's function $\boldsymbol{G}$ as a power series in $\boldsymbol{H}_1$, its ensemble average can be written

$$\langle \boldsymbol{G}\rangle = \boldsymbol{G}_0 + \boldsymbol{G}_0\boldsymbol{\Sigma}\langle\boldsymbol{G}\rangle\,, \tag{29}$$

where $\boldsymbol{G}_0 = \boldsymbol{H}_0^{-1}$ and $\boldsymbol{\Sigma}$ is a self-energy. In the limit $N\to\infty$, the self-consistent Born approximation is exact [9,20,21]. The eigenvalue density can be obtained from $G_{21}(z)$. To study eigenvector correlations, it is necessary to calculate averages of products of $\boldsymbol{G}$'s. In particular, the density

$$D(z_1,z_2) = \delta(z_1-z_2)O(z_1) + O(z_1,z_2) \tag{30}$$

can be written as

$$D(z_1,z_2) = \frac{1}{\pi^2}\frac{\partial}{\partial\overline{z}_1}\frac{\partial}{\partial z_2}\lim_{\eta\to 0}\left\langle\frac{1}{N}\,\mathrm{Tr}\,G_{21}(z_1)G_{12}(z_2)\right\rangle. \tag{31}$$

We therefore calculate $\boldsymbol{R}(z_1,z_2) \equiv \langle\boldsymbol{G}(z_1)\otimes\overline{\boldsymbol{G}}(z_2)\rangle$, which obeys a Bethe-Salpeter equation [20]: writing $\boldsymbol{R}_0(z_1,z_2) = \langle\boldsymbol{G}_0(z_1)\rangle\otimes\langle\overline{\boldsymbol{G}}_0(z_2)\rangle$,

$$\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{R}_0\boldsymbol{\Gamma}\boldsymbol{R}\,. \tag{32}$$

In the limit $N\to\infty$ with $z_1 \ne z_2$, the vertex is simply $\boldsymbol{\Gamma} = \mathrm{diag}(1,\tau,\tau,1)$. Solving Eq. (33) for $\boldsymbol{R}$, we obtain

$$D(z_1,z_2) = -\frac{1}{\pi^2|z_1-z_2|^4} \times \frac{(1-\tau^2)^2 - (1+\tau^2)z_1\overline{z}_2 + \tau(z_1^2+\overline{z}_2^2)}{(1-\tau^2)} \tag{33}$$

for z_1 and z_2 within the support of the density of states, and zero otherwise. Since we have taken $z_1 \ne z_2$, this is simply $O(z_1,z_2)$, and for $\tau = 0$, Eq. (8) is reproduced. For $1-\tau \ll 1$, on the other hand, $O(z_1,z_2)/[d(z_1)\times d(z_2)] \propto 1-\tau$, so that $O(z_1,z_2)/[d(z_1)d(z_2)]$ vanishes in the Hermitian limit $\tau\to 1$, as expected.

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