Phase Jumps near a Phase Synchronization Transition in Systems of Two Coupled Chaotic Oscillators

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Phase synchronization transitions in two different coupled chaotic systems (Rössler and Lorentz) are investigated and shown to be well described by a reduced model of an overdamped periodically driven nonlinear oscillator with a time varying coefficient. In both systems, the phase separation increases with 2π phase jumps below the transition. The scaling rules of the jump near and away from the transition are studied: Near the transition the average interval between two successive jumps follows $\ln\langle l \rangle \sim -(\epsilon_c - \epsilon)^{1/2}$, while away from the transition $\langle l \rangle \sim (\epsilon_t - \epsilon)^{-1/2}$ for both systems. [S0031-9007(98)06625-3]

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In recent years, there has been much interest in understanding complex dynamics that arises in various systems of interacting nonlinear dynamical units. Examples can be found ubiquitously in physical, chemical, biological, and physiological worlds. Coupled nonlinear chemical oscillators [1], populations of social amoebae [2], and neural networks [3] are good examples, just to name a few. One of the exciting scientific quarries on these coupled systems is to understand the coherent dynamical behavior of the coupled system. The most interesting recent development in this regard is the so-called "phase synchronization" (PS) phenomenon observed in systems of two coupled chaotic oscillators [4–6]. Above a critical strength of the coupling, suitably defined phases of two chaotic oscillators lock each other and synchronize, while their amplitudes remain uncorrelated with each other and sustain an irregular motion of their own.

In this Letter, we discuss the physical mechanism for this phenomenon first in a "phase-coherent" Rössler system and later in a "non-phase-coherent" Lorentz system. Both systems show the same mechanism and the same scaling properties near and away from the transition. The PS phenomenon in a coupled Lorentz system is reported for the first time.

The phenomenon was first observed by Rosenblum *et al.* [4] in a numerical simulation on a system of two coupled chaotic Rössler oscillators,

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + \epsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + 0.15y_{1,2}, \\ \dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10), \end{aligned} \tag{1}$$

where $\omega_{1,2}$ is the overall frequency of each chaotic oscillator and ϵ measures the strength of the coupling. Their result is reproduced in our Fig. 1. The PS transition occurs at $\epsilon = 0.0286$ $(= \epsilon_c)$ for the parameter values given in the figure caption. The phase difference (θ) increases monotonically for $\epsilon < \epsilon_c$, while for $\epsilon > \epsilon_c$ it is basically $\pi/2$ with a small amplitude fluctuation of high frequency. Rosenblum *et al.* have identified this transition by studying the spectra of Lyapunov exponents [4]. They have demonstrated that the transition occurs when one of the exponents becomes zero. Nevertheless, the underlying physical mechanism for this transition has not been clearly explained.

In an attempt to elucidate the phenomenon, we have looked more closely near this transition and find that the θ increases with a sequence of 2π jumps below the transition as shown in the Fig. 1 inset. Furthermore, we find another transition at $\epsilon = 0.0276$ $(= \epsilon_t)$: It is realized that the θ increases with an intermittent sequence of 2π jumps for $\epsilon_t \lesssim \epsilon < \epsilon_c$, whereas the θ increases in a nearly periodic sequence of 2π jumps for $\epsilon \lesssim \epsilon_t$. The transitions at ϵ_c and ϵ_t both are continuous. Since the phase desynchronization proceeds with the 2π jumps,

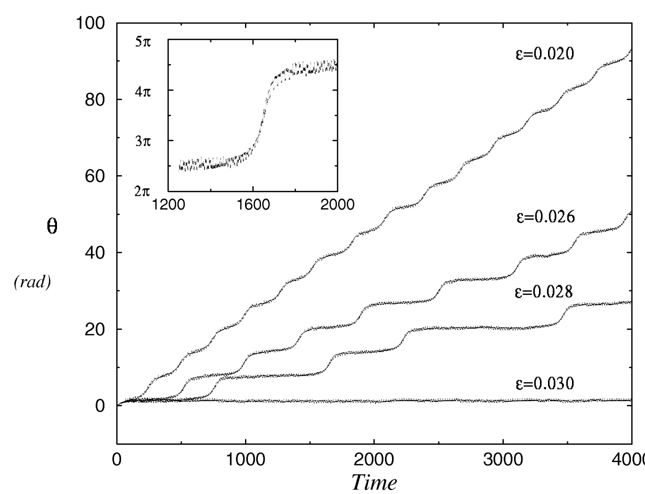

FIG. 1. Time evolutions of phase separation (θ) in a system of two coupled Rössler attractors for various values of ϵ. The phase difference is obtained by $\theta = \phi_1 - \phi_2 = \arctan(y_1/x_1) - \arctan(y_2/x_2)$ [7]. Equation (1) is numerically solved using a fourth-order Runge-Kutta method, $\omega_{1,2} = 1.015$ and 0.985, respectively, and the same values are used throughout this paper. The inset focuses a single 2π jump that appeared in the plot for $\epsilon = 0.028$.

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understanding their physical origin is of an essential importance in characterizing the nature of the transition. Here we provide a physical picture for these phase jumps using a simple model reduced from Eq. (1), and we investigate the scaling properties of them near and away from the transition.

The simplified model is

$$\frac{d\theta}{dt} = (\omega_1 - \omega_2) - \frac{\epsilon}{2}\left(\frac{A_2}{A_1} + \frac{A_1}{A_2}\right)\sin(\theta) \equiv F(\theta)\,, \tag{2}$$

where $\theta = (\phi_1 - \phi_2)$, $A_{1,2} = \sqrt{{x_{1,2}}^2 + {y_{1,2}}^2}$ and $(\omega_1 - \omega_2)$ is the overall frequency mismatch. The θ and $\frac{d\theta}{dt}$ in Eq. (2) should be understood as the average values over the slow time scale originating from the frequency mismatch $(\omega_1 - \omega_2)$. The process leading to Eq. (2) from Eq. (1) was briefly discussed by others to obtain a qualitative estimate of ϵ_c but with no clear proof [8]. In the following, we explicitly demonstrate that the original system [Eq. (1)] with a proper reduction is the same as the simplified model [Eq. (2)], and then we explain the physical mechanism for the observed 2π jumps.

Shown in Figs. 2(a) and 2(b) are numerically computed $\frac{d\theta}{dt}$ as a function of θ for two different values of ϵ. Equation (1) is numerically integrated as explained in the caption of Fig. 1, and θ and $\frac{d\theta}{dt}$ are computed. Those values are averaged over a period of $2\pi/(\omega_1 - \omega_2)$ to remove the fast time-scale dynamics that occur over the time duration of $2\pi/\omega_{1,2}$ in the spirit of the assumption used in obtaining the reduced model [9]. It is quite clear that the functional form of the constructed $\frac{d\theta}{dt}$ closely follows that of $F(\theta)$ given in Eq. (2). This functional form is double checked by computing the right-hand side of Eq. (2). The K $(= \frac{A_2}{A_1} + \frac{A_1}{A_2})$ fluctuates between 2 and 4 periodically with a typical mean value of 2.1.

The 2π jump can be naturally explained by realizing that the reduced model is nothing but a model describing an overdamped particle moving in a "noisy wash-board potential" as shown in Figs. 2(c) and 2(d). There are two factors influencing the sliding of the particle: One is the shape of the potential by itself, and the other is the fluctuating amplitude factor K. If we assume that K is a constant for simplicity (for example, $K = 2$), the system would show a saddle-node bifurcation at $\epsilon = (\omega_1 - \omega_2)$. Above the bifurcation point $[\epsilon > (\omega_1 - \omega_2)]$, the potential would acquire a series of local minima, and the particle would be trapped in one of them permanently. Below the transition, where there is no stable fixed point, the particle would slide 2π-periodically. The qualitative difference in the shapes of $V(\theta)$ is visible in Figs. 2(c) and 2(d).

With the time-varying factor K incorporated in the model, the dynamics of the 2π phase jumps becomes nontrivial. Figure 3(a) with 3(c) illustrates a typical case away from the transition. In this case, the strength of the periodic forcing $(\frac{\epsilon K}{2})$ stays below the bifurcation threshold 0.03 all of the time, and it can be simply regarded as thermal noise as far as the particle trapping (or PS) is concerned. Here, the constant force term

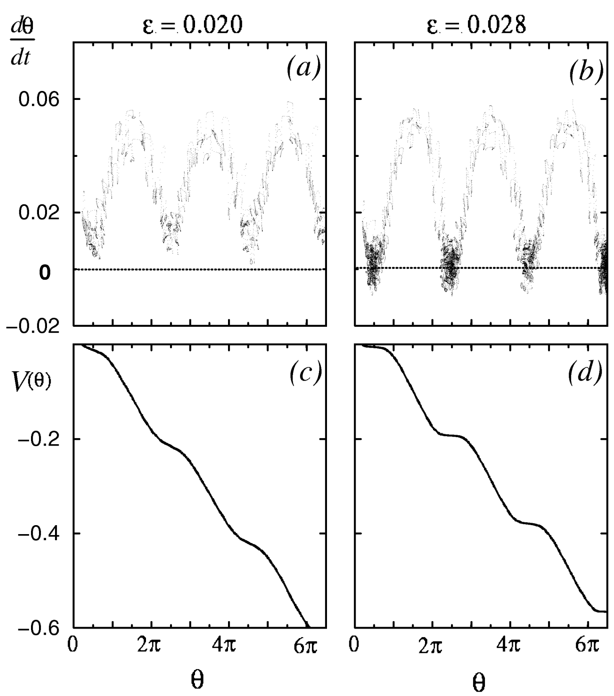

FIG. 2. Numerically computed force $F(\theta) = \frac{d\theta}{dt}$ in (a) and (b), and corresponding potential $V(\theta) = -\int F(\theta)d\theta$ in (c) and (d) as functions of θ. The horizontal dotted lines in (a) and (b) indicate the threshold for a saddle-node bifurcation.

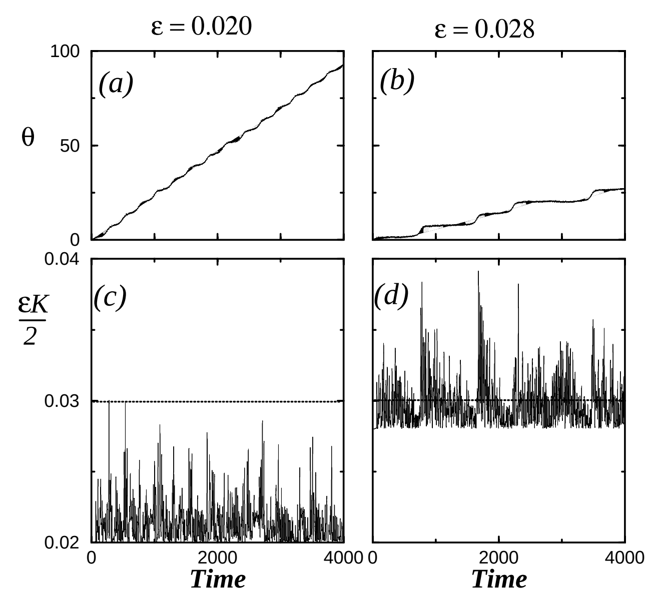

FIG. 3. Temporal evolutions of phase separation (θ) and corresponding nonlinear strength factor $(\epsilon K/2)$, one near and the other away from the onset of PS transition. A saddle-node bifurcation occurs when $\epsilon K/2$ crosses the dotted horizontal line.

$(\omega_1 - \omega_2)$ dominates the dynamics of θ. Figure 3(b) with 3(d) obtained very near the PS transition contrasts the previous case: The $(\frac{\epsilon K}{2})$ stays mostly above 0.03 on the average except for some durations of time, during which a 2π jump occurs. The observed 2π jump is analogous to the phase slippage process corresponding to thermal activation over a high energy barrier in dc Josephson [10]. However, the aperiodic time-varying amplitude factor K is not of thermal nature but follows the deterministic dynamics of the amplitudes. Indeed, the dynamics of phase jumps very near ϵ_c is strongly correlated with the temporal evolution of K.

In an attempt to quantify the qualitative difference in the dynamics of phase separation near and away from the transition, we have computed the probability distribution function of time interval between two successive 2π jumps $P(l)$ and the average time interval $\langle l \rangle$ for various values of coupling constant ϵ. Figure 4(a) shows three $P(l)$s some distance away. Their statistics follow very closely to that of a normal distribution. The bandwidth becomes narrower as the system moves farther away from the transition indicating that the system is more dominated by the constant force term $(\omega_1 - \omega_2)$. This behavior dramatically changes when the system approaches to the close vicinity of the transition point as shown in Fig. 4(b). The distribution function no longer retains the symmetry of normal distribution and becomes a Lorentzian shape. The bandwidth becomes much broader as ϵ moves closer toward ϵ_c [notice the different scales in Figs. 4(a) and 4(b)].

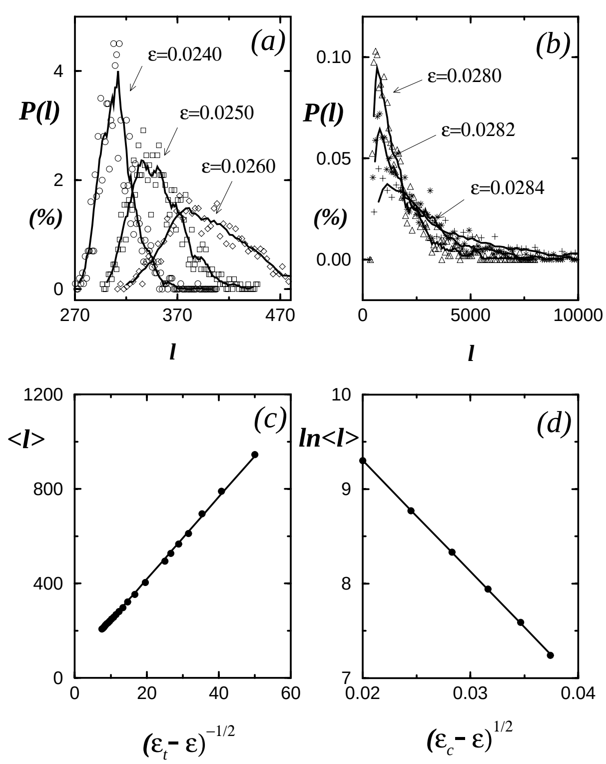

FIG. 4. Probability distribution function $P(l)$ of the 2π jump interval (l) in (a) and (b); $\langle l \rangle$ vs $(\epsilon_t - \epsilon)^{-1/2}$ in (c); and $ln\langle l \rangle$ vs $(\epsilon_c - \epsilon)^{1/2}$ in (d). All are computed for a time duration of 1000 numbers of 2π jumps. The solid lines in (a) and (b) are running averages on five neighbors of raw data. The solid lines in (c) and (d) are straight lines fitting to the numerically obtained data with $\epsilon_t = 0.0276$ and $\epsilon_c = 0.0286$, respectively.

The qualitative difference in $P(l)$s near and away from the transition is also reflected in the different scaling behavior of $\langle l \rangle$ as shown in Figs. 4(c) and 4(d). Upon the decreasing sequence of ϵ from ϵ_c, $\langle l \rangle$ gradually approaches to an asymptotic value of $\langle l \rangle_0 = 209.0$ [$= 2\pi/(\omega_1 - \omega_2)$] at $\epsilon = 0$. During the decrease, there are two distinct regimes in which $\langle l \rangle$ obeys different scaling rules: $\langle l \rangle \sim (\epsilon_t - \epsilon)^{-1/2}$ for $\epsilon \lesssim \epsilon_t$ and $\ln\langle l \rangle \sim -(\epsilon_c - \epsilon)^{1/2}$ for $\epsilon_t \lesssim \epsilon < \epsilon_c$. The scaling rule $\langle l \rangle \sim (\epsilon_t - \epsilon)^{-1/2}$ together with the corresponding $P(l)$ of Fig. 4(a) for $\epsilon \lesssim \epsilon_t$ is the same of type-I intermittency [11]. In this regime, the dynamics of K could be considered as random noise playing the role of chaotic reinjection for a type-I intermittency to occur. The intermittent behavior is weak since the system is away from the saddle-node bifurcation. On the other hand, for the regime $\epsilon_t \lesssim \epsilon < \epsilon_c$, the scaling rule is different and the associated $P(l)$ is a Lorentzian distribution that is also distinguished from the one of type-I intermittency. The phenomenon of exponentially rare 2π phase jumps very near the transition is consistent with the statistical law of the eyelet intermittency found in a circle map coupled to a perturbed tent map that was recently developed to model the PS phenomenon [12]. When their study is translated to our analysis, $\frac{\epsilon K}{2}$ should stay below the saddle-node bifurcation line for some minimum duration for a 2π jump to occur. This is clearly seen in our Fig. 3(d).

All of these observations appear to be quite general and applicable to any coupled chaotic system as long as the involved chaotic attractor keeps some degree of "phase coherence." The phase coherence of a chaotic attractor means that a suitably defined phase increases steadily in time. In this sense, the Rössler system discussed above is an example of a perfect phase coherent system, while the Lorentz system discussed next is not a phase coherent system.

With a system of coupled Lorentz chaotic oscillators we qualitatively find the same properties as shown in Fig. 5. Utilizing the reflection symmetry ($x \leftrightarrow -x$ and $y \leftrightarrow -y$), a new variable $u = \sqrt{x^2 + y^2}$ is defined, and a phase can be suitably defined on a u-z plane with the origin at the unstable fixed point in the middle. In this representation, the Lorentz system becomes almost phase coherent but with some occasional retreats as shown in the bottom left of Fig. 5(a). Nevertheless, the similarity between Fig. 5(b) and Fig. 1 is quite clear. The phase desynchronization in Fig. 5(b) also proceeds with an intermittent sequence of 2π phase jumps that becomes more frequent away from the transition. The scaling rules of $\langle l \rangle$ near and away from the transition are the same as in the Rössler system [Figs. 5(c) and 5(d)]. The phase increase away from the transition is not quite periodic, since the fluctuation of K is much more drastic than it is in the Rössler system (not shown).

In summary, we have shown that the PS phenomenon in a system of two coupled Rössler attractors can be well approximated by a single first-order equation [Eq. (2)] for θ with the time-varying amplitude factor K. The same physical picture is also applicable to the coupled Lorentz system. The different scaling rules of $\langle l \rangle$ near and away from the PS transition are found and the continuous transition at ϵ_t is identified for the first time and discussed in connection with the dynamics of K. The PS phenomenon demonstrated in the coupled Lorentz system is significant, since the system retains some degree of phase noncoherence.

We conclude our study by listing several important issues left for future investigation. First, a general framework based on different time scales would be developed for the reduction of the original systems to the type of Eq. (2). Second, the PS phenomenon would be extended to a larger context of frequency locking of m:n (m and n, incommensurate integers): The phase synchronization is only a particular case of a winding number, $\Omega = m/n = 0/1$ [13,14]. Third, the effect of phase incoherence in the PS phenomenon would be studied. Finally, the PS phenomenon in a system of two coupled chaotic attractors can be generalized to a population of attractors. The coherent spiral waves recently observed in a coupled chaotic system by Goryachev and Kapral can be understood in this venue of thought [15]. Also, Osipov *et al.* recently reported PS effects in a lattice of a nonidentical Rössler oscillator in relation to clustering dynamics and defect dynamics [16].

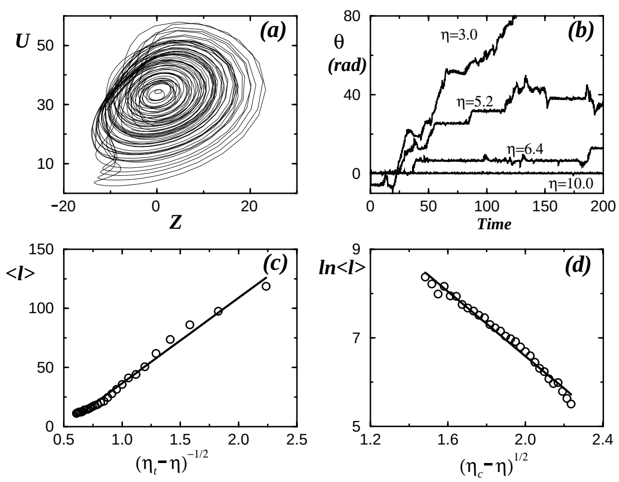

FIG. 5. Phase synchronization phenomenon in a system of two coupled Lorentz attractors: $\dot{x}_{1,2} = 10.0(y_{1,2} - x_{1,2}) + \eta(x_{2,1} - x_{1,2})$; $\dot{y}_{1,2} = (36.5 + \gamma_{1,2})x_{1,2} - y_{1,2} - x_{1,2}z_{1,2}$; and $\dot{z}_{1,2} = -3.0z_{1,2} + x_{1,2}y_{1,2}$ ($\gamma_{1,2} = 1.5$ and -1.5, respectively), showing: (a) one of two coupled Lorentz attractors projected on the u-z plane; (b) θ vs time for different values of η; (c) $\langle l \rangle$ vs $(\eta_t - \eta)^{-1/2}$; and (d) $\ln\langle l \rangle$ vs $(\eta_c - \eta)^{1/2}$. $\eta_t = 6.7$ and $\eta_c = 12.0$.

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Note added.—After the submission of this Letter, an independent study on the phenomenon of phase synchronization appeared [17]. Their model, a modified Rössler system driven by a periodic forcing, also shows the presence of 2π phase jumps. The scaling rule of exponentially rare phase jumps is reported and viewed as the unstable-unstable pair bifurcation crisis.

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