Theory of Heating of the Quantum Ground State of Trapped Ions

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Using a displacement operator formalism, I analyze the depopulation of the vibrational ground state of trapped ions. Two heating times, one characterizing short time behavior and the other long time behavior, are found. The long time behavior is analyzed for both single and multiple ions, and a formula for the relative heating rates of different modes is derived. The possibility of correction of heating via the quantum Zeno effect, and the exploitation of the suppression of heating of higher modes to reduce errors in quantum computation, is considered. [S0031-9007(98)06590-9]

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Individual or multiple ions can be confined in a radiofrequency Paul trap and using sophisticated laser techniques cooled to the quantum mechanical ground state [1,2]. Such systems allow experimental tests of fundamental principles, such as the preparation and measurement of nonclassical motion states [3] or the observation of superradiance [4], and are therefore of great current interest. Furthermore, important technological applications for such systems, such as very precise frequency standards [5] or the practical implementation of quantum computation [6], have recently attracted considerable attention. The accuracy of trapped ion frequency standards derives in part from the reduction of Doppler broadening of atomic transitions due to trapping and cooling. In a trapped ion quantum computer, information will be transferred between different ions (which constitute the quantum bits, or "qubits" of the computer) using quantum states of the collective motion of the ions in the harmonic confining potential; if these motional quantum states become degraded, the information will be lost. Thus the maintenance of the trapped ions at a very low temperatures is of great importance for both of these applications. In particular, of the many practical difficulties for ion trap quantum computers, one of the most important is the very fragile nature of the motional ground state. In this Letter I present a theoretical analysis of the depopulation of the motional ground state of ions due to ambient classical electromagnetic fields, which can be loosely characterized as heating (although it should be stressed that relaxation to a thermal distribution is *not* considered here).

Various analyses of decoherence mechanisms in ion trap quantum computers have been carried out [7–11]. Such effects as spontaneous emission [8–10] or dephasing due to the ions' zero-point motion [7] have been considered. Also the relaxation of an ion to a thermal state has been considered using a perturbation method, valid for describing long time behavior [12]; other work has been done on the effects of laser amplitude and phase stability [13]. In this Letter I adopt a different approach than that normally used in analysis of decoherence problems. When the ambient electromagnetic fields can be treated classically, the equations of motion reduce to a solvable system. One can dispense with the density matrix/master equation formalism, using instead the interaction picture to solve exactly for the wave function describing the state of the ion, thereby obtaining a formula for the population remaining in the ground state. An average of this population over the field fluctuations yields simple expressions for the depopulation.

In the interaction picture the Hamiltonian describing the interaction of a single ion of mass M with a *classical* electric field $E(t)$ is given by the following formula:

$$\hat{H} = i\hbar[u(t)\hat{a}^{\dagger} - u^{*}(t)\hat{a}], \tag{1}$$

where $u(t) = ieE(t)\exp(i\omega_0 t)/\sqrt{2M\hbar\omega_0}$, e being the ion charge and $\hbar$ Planck's constant divided by 2π. I am considering only the motion of the ion along the weak confinement axis of an anisotropic trap, of the sort suitable for quantum computation; thus $E(t)$ represents the component of the electric field along that axis. The harmonic binding potential is characterized by the angular frequency ω_0. The operators $\hat{a}$ ($\hat{a}^{\dagger}$) are the zero time annihilation (creation) operators for the harmonic motion of the ion in the harmonic well.

The dynamics of such a driven quantum harmonic oscillator can be solved exactly [14,15]. The wave function at some instant t is related to the initial wave function by the following expression:

$$|\psi(t)\rangle = \exp[i\phi(t)]\hat{D}[v(t)]|\psi(0)\rangle, \tag{2}$$

where $\hat{D}[v] = \exp[v\hat{a}^{\dagger} - v^{*}\hat{a}]$ is the displacement operator, $\phi(t)$ is a phase factor (which turns out to be unimportant for the current problem), and the amplitude $v(t)$ is given by the formula

$$v(t) \equiv \int_0^t u(t')dt' = \frac{ie}{\sqrt{2M\hbar\omega_0}}\int_0^t E(t')\exp(i\omega_0 t')dt'. \tag{3}$$

As a figure of merit, let us introduce an average fidelity of the ground state, defined by the following formula:

$$F(t) = \langle |\langle \psi(t)|\psi(0)\rangle_q|^2\rangle_f \,, \tag{4}$$

where $\langle\cdots\rangle_q$ denotes a quantum mechanical average, and $\langle\cdots\rangle_f$ an average over an ensemble of realizations of the classical random field $E(t)$. If one assumes that the ion is initially in the ground state, then its state evolves into a coherent state of amplitude $v(t)$. Thus the probability of remaining in the ground state can be found in closed form, and the fidelity of the ground state is given by the formula

$$F(t) = \langle \exp[-|v(t)|^2]\rangle_f \,. \tag{5}$$

If one assumes Gaussian statistics for the classical random field $E(t)$ [16] (which can be justified by assuming that the field is due to many uncorrelated random sources, and then invoking the central limit theorem), the average over the field ensemble can be determined by performing an integration using the appropriate probability distribution. The result is as follows:

$$\begin{aligned} F(t) = \; & [1 + 2\langle |v(t)|^2\rangle_f \\ & + (\langle |v(t)|^2\rangle_f)^2 - |\langle v(t)v(t)\rangle_f|^2]^{-1/2}. \end{aligned} \tag{6}$$

An alternative measure of the heating is the mean excitation number, defined by

$$\bar{n}(t) = \langle [\langle \psi(t)\,|\hat{a}^\dagger \hat{a}|\psi(t)\rangle_q]\rangle_f \,. \tag{7}$$

Using Eq. (2) and Eq. (11.3-13) of Ref. [15], this can be rewritten as

$$\begin{aligned} \bar{n} &= \langle [\langle 0|\hat{n} + v^*(t)\hat{a} + \hat{a}^\dagger v(t) + |v(t)|^2|0\rangle_q]\rangle_f \\ &= \langle |v(t)|^2\rangle_f \,. \end{aligned} \tag{8}$$

The correlation functions appearing in Eqs. (6) and (8) can be evaluated using Eq. (3). Using the symmetry property of the autocorrelation function, one obtains

$$\langle |v(t)|^2\rangle_f = \Omega^2 t^2 \int_0^1 (1-x)\gamma_E(xt)\cos(x\omega_0 t)\,dx \,, \tag{9}$$

$$\begin{aligned} \langle v(t)^2\rangle_f = \; & \frac{\Omega^2 t}{\omega_0}\exp(i\omega_0 t) \\ & \times \int_0^1 \gamma_E(xt)\sin[(1-x)\omega_0 t]\,dx \,, \end{aligned} \tag{10}$$

where $\gamma_E(\tau) = \langle E(t+\tau/2)E(t-\tau/2)\rangle_f/\langle E(t)^2\rangle_f$ is the degree of correlation of the field $E(t)$ (which is real) and the characteristic transition rate Ω is given by $\Omega = \sqrt{e^2\langle E(t)^2\rangle_f/M\hbar\omega_0}$. Since I have implicitly assumed that $E(t)$ is stationary, $\langle E(t)^2\rangle_f$ is independent of time.

For an exponential degree of correlation given by $\gamma_E(\tau) = \exp(-|\tau|/T)$, the integrals given in Eqs. (9) and (10) can be evaluated in closed form:

$$\begin{aligned} \langle |v(t)|^2\rangle_f &= \bar{n}(t) \\ &= \frac{T}{\tau_1}[\exp(-t/T)\cos(\omega_0 t + 2\phi) \\ &\quad - \cos(2\phi) + t/T] \,, \end{aligned} \tag{11}$$

$$\begin{aligned} \langle v(t)^2\rangle_f = \; & \frac{T}{\tau_1}\exp(i\omega_0 t)[\exp(-t/T)\sin(\phi) \\ & + \sin(\omega_0 t - \phi)] \,, \end{aligned} \tag{12}$$

where $\tan\phi = \omega_0 T$ and the heating time τ_1 is given by the formula

$$\frac{1}{\tau_1} = \left(\frac{e^2\langle E(t)^2\rangle_f}{M\hbar\omega_0}\right)\frac{T}{1+\omega_0^2 T^2} \,. \tag{13}$$

Examples of the fidelity calculated using these results are shown in Fig. 1. These show that revivals of the ground state populations can occur when the heating field has both a low amplitude and a long coherence time.

Two limiting cases are of interest, namely, the behavior for short times and for very long times. For short times (i.e., $t \ll T, 1/\omega_0$) the mean excitation number and fidelity are given by

$$\bar{n}(t) \approx 1 - F(t) = \left(\frac{1+\omega_0^2 T^2}{2T\tau_1}\right)t^2 + O(t^3) \,. \tag{14}$$

This result, i.e., that for short times the "decay" of the ground state population is nonexponential, allows the possibility of maintaining the ion in its ground state via the *quantum Zeno effect* [17,18]. If repeated measurements of the ground state population can be made on time scales much shorter than $1/\omega_0$, then the ion will in principle remain in the ground state for a much longer time. However, as the quadratic behavior persists for a short time only (see Fig. 1), such a technique would be very difficult to implement. Furthermore, by invoking the time-energy uncertainty principle, one can see that measurements carried out on time scales much less than $1/\omega_0$ will result in excitation of higher number

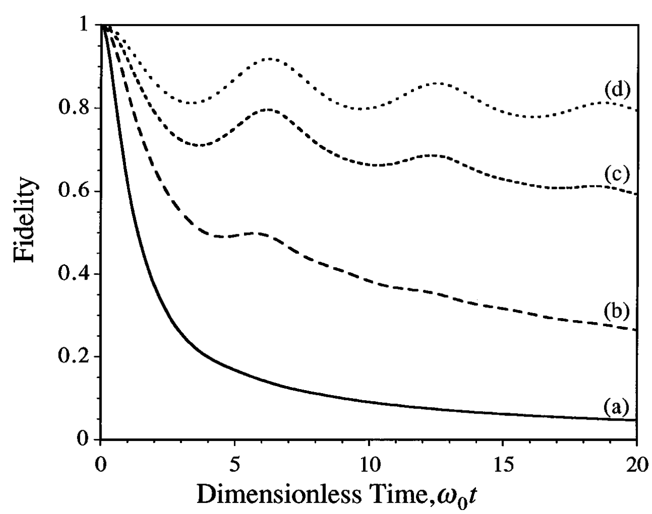

FIG. 1. The fidelity of the ground state as a function of time, illustrating the results given in Eqs. (6), (11), and (12). The parameters used were as follows: curve (*a*) $\omega_0 T = 1$, $\omega_0\tau_1 = 1$; curve (*b*) $\omega_0 T = 1$, $\omega_0\tau_1 = 8.5$; curve (*c*) $\omega_0 T = 1$, $\omega_0\tau_1 = 41$; and curve (*d*) $\omega_0 T = 1$, $\omega_0\tau_1 = 128.5$.

states. Also this technique will not preserve the coherence of superpositions of different phonon number states, which occur during the implementation of gate operations. Hence it is unlikely that the Zeno effect will be a practical method of error correction.

One can find asymptotic expressions for $F(t)$ and $\bar{n}(t)$ in the long time limit $t \gg T, 1/\omega_0$:

$$\bar{n}(t) \sim \frac{1}{\tau_1}(t - t_0), \tag{15}$$

$$F(t) \sim \frac{\tau_1}{t}, \tag{16}$$

where $t_0 = T(1 - \omega_0^2 T^2)/(1 + \omega_0^2 T^2)$. These results are in qualitative agreement with that obtained earlier by Lamoreaux [12] using a perturbative density matrix approach.

We can estimate the magnitude of the ambient electric field which causes the heating of the ions by assuming that the coherence time T is roughly equal to $1/\omega_0$. Equation (13) then implies that the rms electric field strength is given by the formula

$$E_{\rm rms} = \sqrt{M\hbar\omega_0^2/e^2\tau_1}\,. \tag{17}$$

Using data from the mercury ion experiment described in Ref. [1] [i.e., $M = 3.29 \times 10^{-25}$ kg, $\omega_0 = (2\pi)4.66$ MHz], a heating time $\tau_1 = 135$ ms implies $E_{\rm rms} \approx 3 \times 10^{-3}\ {\rm V\,m^{-1}}$.

Let us now consider the case of multiple ions confined in a linear configuration. Because the ions are interacting via the Coulomb force, their motion will be strongly coupled. The small amplitude fluctuations are best described in terms of normal modes, each of which can be treated as an independent harmonic oscillator [19]. If there are N ions in the trap, there will be a total of N such modes. I will number these modes in order of increasing resonance frequency, the lowest ($p = 1$) mode being the center of mass mode, in which the ions oscillate as if rigidly clamped together. In the quantum mechanical description, each mode is characterized by creation and annihilation operators $\hat{a}_p^\dagger$ and $\hat{a}_p$ (where $p = 1, \ldots, N$). The Hamiltonian in this case is given by the expression

$$\hat{H} = i\hbar \sum_{p=1}^{N} [u_p(t)\hat{a}_p^\dagger - u_p^*(t)\hat{a}_p], \tag{18}$$

where

$$u_p(t) = \frac{ie}{\sqrt{2M\hbar\omega_0\sqrt{\mu_p}}} \sum_{n=1}^{N} E_n(t) b_n^{(p)} \exp(i\sqrt{\mu_p}\,\omega_0 t)\,. \tag{19}$$

In Eq. (19), $E_n(t)$ is the electromagnetic field at the nth ion of the string, $b_n^{(p)}$ is the nth element of the pth normalized eigenvector of the ion coupling matrix [19], μ_p being its eigenvalue. Again the evolution of the state of the ions can be solved exactly,

$$|\psi(t)\rangle = \exp[i\Phi(t)] \prod_{p=1}^{N} \hat{D}_p[v_p(t)]\,|\psi(0)\rangle, \tag{20}$$

where $\hat{D}_p[v_p] = \exp[v_p\hat{a}_p^\dagger - v_p^*\hat{a}_p]$ is the displacement operator for the pth mode, and

$$v_p(t) = \int_0^t u_p(t')\,dt'. \tag{21}$$

As before, one can find a closed form expression for the fidelity of the ground state of the string of ions. For multiple ions, the mean excitation number is given by a formula analogous to Eq. (16), where the characteristic decay time is given by

$$\tau_N = \tau_1 \left[\sum_{p=1}^{N} \sum_{m,n=1}^{N} \frac{b_m^{(p)} b_n^{(p)}}{\sqrt{\mu_p}}\, \gamma_{mn} \right]^{-1}, \tag{22}$$

γ_{mn} being the degree of coherence of the field at the positions of ions n and m evaluated for zero time delay [15] (the field having been assumed to be cross-spectrally pure [20,21]), and the coherence time T has been assumed to be much less than $1/\omega_0$. In the coherent limit ($\gamma_{mn} = 1$), the formula for τ_N reduces to the following simple expression:

$$\tau_N = \frac{\tau_1}{N}, \tag{23}$$

while in the incoherent limit ($\gamma_{mn} = \delta_{m,n}$), τ_N is given by the formula

$$\tau_N = \tau_1 \left[\sum_{p=1}^{N} \frac{1}{\sqrt{\mu_p}} \right]^{-1}. \tag{24}$$

The sum can be worked out from the eigenvalues μ_p, which must be determined numerically in general. The results are shown in Fig. 2. The separation of ions

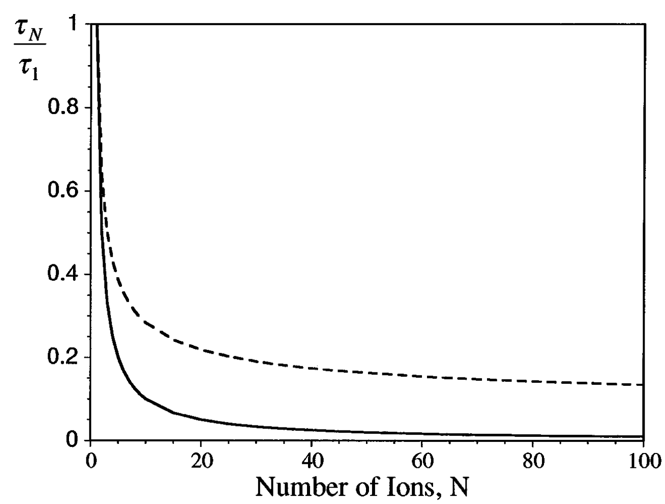

FIG. 2. The heating time of the ground state for different numbers of ions, in the spatially coherent (plain curve) and spatially incoherent (dashed curve) limits for the ambient electric field.

is generally of the order of a few tens of microns. Electromagnetic radiation with frequencies of the order of MHz has wavelengths of hundreds of meters, so that the coherence length of the radiation can be expected to be much larger than the separation of the ions. Therefore it is the coherent limit that is the important one, at least for small numbers of ions.

One can also consider the mean excitation numbers of the different modes. These quantities are given by expressions analogous to Eq. (8), with the heating time of the pth mode, when there are N ions in the string, given by

$$\tau_{N,p} = \frac{\sqrt{\mu_p}}{\sum_{m,n=1}^{N} b_m^{(p)} b_n^{(p)} \gamma_{mn}} \tau_1 . \tag{25}$$

In the coherent and incoherent limits, the heating times of the different modes are given by

$$\tau_{N,p} = \begin{cases} \tau_1/N & p = 1 \\ \infty & p > 1 \end{cases} \text{ coherent} \tag{26}$$

$$\tau_{N,p} = \sqrt{\mu_p}\,\tau_1 \text{ incoherent.} \tag{27}$$

The heating times for the coherent case is potentially a very important result. Only the lowest ($p = 1$) center of mass mode will be heated up by spatially coherent fields. Thus the state of the ion oscillations is given by

$$|\psi(t)\rangle = \exp i\phi(t)\{|v_1(t)\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes \cdots \otimes |0\rangle_N\}, \tag{28}$$

the modes other than the center of mass mode remaining in their ground states. Since (with a few exceptions) each ion couples with each phonon mode, any one of these modes can be utilized as a quantum information bus in an exactly analogous manner to the way the center of mass mode was utilized in Cirac and Zoller's original proposal. Thus it will be possible to perform quantum logic operations without degradation due to heating, simply by utilizing these higher modes, which have much slower heating rates [22]. These modes can be selectively excited or deexcited by tuning the addressing laser to the appropriate mode frequency, and controlling the pulse duration by the appropriate amount, although care must be exercised in this procedure because of the variations in coupling strength between different ions to the different modes (for details, see Ref. [19]).

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