Relaxed States of a Magnetized Plasma with Minimum Dissipation

B. Dasgupta,[1] P. Dasgupta,[2] M. S. Janaki,[1] T. Watanabe,[3] and T. Sato[3]
[1]*Saha Institute of Nuclear Physics I/AF, Bidhannagar, Calcutta 700 064, India*
[2]*Department of Physics, University of Kalyani, Kalyani 741235, India*
[3]*Theory and Computer Simulation Center, National Institute for Fusion Science, Toki, Gifu 509-52, Japan*
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The relaxed state of a slightly resistive and turbulent magnetized plasma is obtained by invoking the principle of minimum dissipation, which leads to $\nabla \times \nabla \times \nabla \times \mathbf{B} = \Lambda \mathbf{B}$. A solution of this equation is accomplished using the analytic continuation of the Chandrasekhar-Kendall eigenfunctions in the complex domain. The new features of this theory show (i) that a single fluid can relax to an MHD equilibrium which can support a pressure gradient even without a long-term coupling between mechanical flow and magnetic field, and (ii) field reversal in states that are not force free. [S0031-9007(98)07284-6]

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In the well-known theory of relaxation of magnetoplasma, Taylor [1] proposed that the process of relaxation is governed by the principle of minimum total magnetic energy and invariance of total (global) magnetic helicity $K = \int_V \mathbf{A} \cdot \mathbf{B} dV$ where the integration is over the entire volume, the latter being the most significant invariant in the theory of relaxation. Accordingly, the relaxed state of a magnetoplasma satisfies the corresponding Euler-Lagrange equation,

$$\nabla \times \mathbf{B} = \lambda \mathbf{B}, \tag{1}$$

with constant λ, and, consequently, is a force-free state. Taylor's theory is quite successful in explaining a number of experimental results, including those of reversed field pinch (RFP). However, relaxed states as envisaged by Taylor, have only zero pressure gradient.

Extensive numerical works by Sato and his collaborators have established [2,3] the existence of self-organized states with finite pressure, i.e., these states are governed by the magnetohydrodynamic force balance relation, namely, $\mathbf{j} \times \mathbf{B} = \nabla p$, rather than $\mathbf{j} \times \mathbf{B} = 0$. Recently, it has been demonstrated both by numerical simulation [4] and by experiments [5] that the counterhelicity merging of two spheromaks can produce a field-reversed configuration (FRC). The FRC has zero toroidal magnetic field, and the plasma is confined entirely by poloidal magnetic field. It has a finite pressure with a relatively high value of β. It may be concluded that FRC, with its nonzero perpendicular component of current, is a relaxed state and it is a distinctly non-force-free state. From the point of view of plasma relaxation, the formation of FRC through the counterhelicity merging of two spheromaks is a unique process where a non-force-free state emerges from the fusion of two Taylor states. The conclusion is that there exists a general class of relaxed states which are not always force free, and Taylor's force-free states constitute only a subclass of this wider class. While Taylor states do not support any pressure gradient, equilibrium obtained from the principle of minimum energy accommodates pressure gradients only in the presence of flow. Several attempts [6–8] have been made in the past to obtain relaxed states which could support finite pressure gradient, a large number of them making use of the coupling of the flow with magnetic field [9–12].

The principle of "minimum rate of entropy production," formulated by Prigogine and others [13], is believed to play a major role in many problems of irreversible thermodynamics. Dissipation, along with nonlinearity, is ubiquitous in systems which evolve towards self-organized states. Another closely related concept, the principle of minimum dissipation rate, was used for the first time by Montgomery and Phillips [14] in an MHD problem to understand the steady state profiles of RFP configuration under the constraint of a constant rate of supply and dissipation of helicity together with the usual physical boundary conditions for a conducting wall. It may be pointed out that the principle of minimum dissipation was also discussed by Chandrasekhar and Woltjer [15] in a sequel to the complete general solution of the force-free equation by Chandrasekhar and Kendall [16]. The minimum dissipation rate hypothesis was later used by a number of authors [17,18] to predict the current and magnetic field profiles of driven dissipative systems.

This paper deals with the question of determining the field configurations assumed by a magnetofluid in a relaxed state by maintaining that the relaxation is governed by the principle of minimum rate of energy dissipation. It is our conjecture that relaxed states could be characterized as the states of minimum dissipation rather than states of minimum energy. The novel feature of our work is to show that it is possible for a *single* fluid to relax to an MHD equilibrium with a magnetic field configuration which can support pressure gradient, even without a long-term coupling between the flow and the magnetic field. In a recent work, Steinhauer *et al.* [12] claimed that single fluid MHD theory can admit only a force-free state, and

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one needs to take recourse to a two fluid theory so that electromechanical coupling produces pressure gradient and a non-force-free state. Our work establishes that none of these requirements need be satisfied to obtain a relaxed state of the desired kind.

In what follows we derive the Euler-Lagrange equation from a variational principle with minimum energy dissipation and conservation of total magnetic helicity, solve the equation in terms of the analytically continued Chandrasekhar-Kendall (CK) eigenfunctions, discuss the important role played by the boundary conditions, and present our results for the flux, field reversal parameter, pinch parameter, and pressure profile. The results of our theory which predict non-force-free relaxed states are closer to the experimentally observed [19] RFP configurations than Taylor's theory.

We consider a closed system of an incompressible, resistive magnetofluid, without any mean flow velocity, described by the standard MHD equations in the presence of a small but finite resistivity η. In the absence of any externally imposed electric fields, the Ohmic dissipation rate R is itself a time varying quantity. However, it is possible to find constraints that are better preserved than the rate of energy dissipation. In this case, helicity still serves to hold as a good constraint as it decays at a time scale much slower in comparison to the decay time scale of the rate of energy dissipation as is evident from the simulation works of Zhu *et al.* [3]. It can be easily shown that the decay rate of dissipation ($\dot{R}$) is $O(1)$ at scale lengths for which $k \approx S^{1/2}$. But, at these scale lengths, helicity dissipation ($\dot{K}$) is only $O(S^{-1/2}) \ll 1$, where S is the ratio of resistive diffusion time to Alfven time. Thus, we may expect that, in the presence of small scale turbulence with $S \gg 1$, the rate of energy dissipation decays at a faster rate than helicity.

We therefore minimize the ohmic dissipation $R = \int \eta \mathbf{j}^2\, dV$ subject to the constraints of helicity $\int \mathbf{A} \cdot \mathbf{B}\, dV$. The variational equation is given by

$$\delta \int (\eta \mathbf{j}^2 + \overline{\lambda} \mathbf{A} \cdot \mathbf{B})\, dV = 0\,, \tag{2}$$

where $\overline{\lambda}$ is Lagrange's undetermined multiplier. The variation can be shown to lead to the Euler-Lagrange equation

$$\nabla \times \nabla \times \nabla \times \mathbf{B} = \Lambda \mathbf{B}\,, \tag{3}$$

where $\Lambda = \overline{\lambda}/\eta$ is a constant. The surface terms in the equation vanish if we consider the boundary condition $\delta \mathbf{A} \times \mathbf{n} = 0$ as well as $\mathbf{j} \times \mathbf{n} = 0$, which is the physical boundary condition we will impose in the problem.

We emphasize that Eq. (3) is a general equation which embraces the Woltjer-Taylor equation [i.e., Eq. (1)] as a special case. The solution of Eq. (3) can be constructed using the CK eigenfunctions. The CK solution [16] of the equation $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ can be written in cylindrical coordinates as

$$\mathbf{B}(\mu, m, k) = \lambda \nabla \Phi \times \nabla z + \nabla \times (\nabla \Phi \times \nabla z)\,, \tag{4}$$

where $\Phi = J_m(\mu r) \exp[i(m\theta + kz)]$ with $\lambda^2 = \mu^2 + k^2$. Here, J_m is a Bessel function of order m, and the value of μ in the argument is determined from the boundary condition at $r = a$, which is given as $(\hat{\mathbf{n}} \cdot \mathbf{B})_{r=a} = 0$.

The actual physical significance of Eq. (3) could be obtained by exploring such solutions which are *not* at the same time solutions of the Woltjer-Taylor equation, the latter always leading to a restricted class of relaxed states, namely, which are force free. This requires the general solutions to the former equation which we are able to construct by taking a linear combination of CK eigenfunctions with complex λ. For real values of λ (and hence of μ and k), the operator $(\nabla \times)$ has been proved to be self-adjoint, but not so in the larger space spanned by the analytically continued CK solutions.

We introduce the complex parameters

$$\mu_n = [(\mu^2 + k^2) \exp(4n\pi i/3) - k^2]^{1/2}, \qquad n = 1, 2 \tag{5}$$

so that $\mu_n^2 + k^2 = \lambda^2 \omega^{2n}$, $\omega = \exp(2\pi i/3)$, and define

$$\begin{aligned} \mathbf{B}_1 &= \mathbf{B}(\mu, m, k) = \lambda \nabla \Phi \times \nabla z + \nabla \times (\nabla \Phi \times \nabla z)\,, \\ \mathbf{B}_2 &= \mathbf{B}(\mu_1, m, k) = \lambda \omega \nabla \Phi_1 \times \nabla z + \nabla \times (\nabla \Phi_1 \times \nabla z)\,, \\ \mathbf{B}_3 &= \mathbf{B}(\mu_2, m, k) = \lambda \omega^2 \nabla \Phi_2 \times \nabla z + \nabla \times (\nabla \Phi_2 \times \nabla z)\,. \end{aligned} \tag{6}$$

In the last two expressions above, Φ_1 and Φ_2 are obtained from Φ by replacing μ by μ_1 and μ_2, respectively.

A solution of Eq. (3) can now be obtained as a linear combination of $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$:

$$\mathbf{B} = \alpha_1 \mathbf{B}_1 + \alpha_2 \mathbf{B}_2 + \alpha_3 \mathbf{B}_3\,, \tag{7}$$

where α_i are constants, with at least two of them nonzero. It can be easily demonstrated that the expression for $\mathbf{B}$ given in (7) is a solution of Eq. (3) with $\Lambda = \lambda^3$.

A reasonable boundary condition is to assume a perfectly conducting wall, so that

$$\mathbf{B} \cdot \mathbf{n} = 0, \qquad \mathbf{j} \times \mathbf{n} = 0 \quad \text{at } r = a\,. \tag{8}$$

The boundary conditions given by Eq. (8) suffice to fix the arbitrary constants

$$\frac{\alpha_2}{\alpha_1} = -\frac{\omega^2 (B_{1\theta} B_{2z}^* - B_{2\theta}^* B_{1z})|_{r=a}}{(B_{2\theta} B_{2z}^* - B_{2\theta}^* B_{2z})|_{r=a}}\,, \tag{9}$$

$$\alpha_3 = \alpha_2^* . \tag{10}$$

If we relax the boundary condition $\mathbf{j} \times \mathbf{n} = 0$, the constants α_2 and α_3 become free, and Taylor states can be accommodated as solutions of Eq. (3). However, for nontrivial values of the constants α_i, the magnetic fields at the boundary $r = a$ have to obey the following relation:

$$2B_{1r}\,\mathrm{Im}(B_{2\theta}B_{2z}^*) - 2B_{1\theta}\,\mathrm{Im}(\omega^2 B_{2r}B_{2z}^*) + 2B_{1z}\,\mathrm{Im}(\omega^2 B_{2r}B_{2\theta}^*) = 0 . \tag{11}$$

From Eq. (6), it is evident that B_2 and B_3 are complex conjugates of each other. This, together with the relations obtained in Eq. (10), shows that the magnetic field given by Eq. (7) is a real field. We also list the following expressions for the $m = 0, k = 0$ state (cylindrically symmetric state) obtained from Eqs. (4)–(7):

$$B_r = 0 ,$$

$$B_\theta = \lambda^2\alpha_1\left[J_1(\lambda r) + 2\,\mathrm{Re}\left(\frac{\alpha_2}{\alpha_1}\,\omega^2 J_1(\lambda\omega r)\right)\right],$$

$$B_z = \lambda^2\alpha_1\left[J_0(\lambda r) + 2\,\mathrm{Re}\left(\frac{\alpha_2}{\alpha_1}\,\omega^2 J_0(\lambda\omega r)\right)\right].$$

For a given value of m and ka, the value of λa can be obtained from the boundary condition given by Eq. (11). It is to be noted that for the cylindrically symmetric state the boundary condition is trivially satisfied and hence does not determine λa. It can be easily proved that the state of minimum dissipation is equivalent to the state of minimum value of Λ. To get the numerical value of λ for $m \neq 0$, we solve Eq. (11) numerically and obtain $\lambda a = 3.11$ and $ka = 1.23$ as the minimum values of λa and ka for the $m = 1$ state.

The only undetermined constant in Eq. (7) is the value of α_1 (the value of the field amplitude) which can be determined by specifying the toroidal flux Φ_z. The $m = k = 0$ state is responsible for nonzero values of toroidal flux which is obtained as

$$\Phi_z = 2\pi\alpha_1\lambda a\left[J_1(\lambda a) + 2\,\mathrm{Re}\left(\frac{\alpha_2}{\alpha_1}\,\omega J_1(\lambda\omega a)\right)\right]. \tag{12}$$

A couple of dimensionless quantities that have proved useful in describing laboratory experiments are the field reversal parameter $F = B_z(a)/\langle B_z\rangle$ and the pinch parameter $\Theta = B_\theta(a)/\langle B_z\rangle$, where $\langle\cdots\rangle$ represents a volume average. After substituting the expressions for $B_z(a)$, etc., we get

$$F = \frac{\lambda a}{2}\,\frac{J_0(\lambda a) + 2\,\mathrm{Re}[(\alpha_2/\alpha_1)\omega^2 J_0(\lambda\omega a)]}{J_1(\lambda a) + 2\,\mathrm{Re}[(\alpha_2/\alpha_1)\omega J_1(\lambda\omega a)]}, \tag{13}$$

$$\Theta = \frac{\lambda a}{2}\,\frac{J_1(\lambda a) + 2\,\mathrm{Re}[(\alpha_2/\alpha_1)\omega^2 J_1(\lambda\omega a)]}{J_1(\lambda a) + 2\,\mathrm{Re}[(\alpha_2/\alpha_1)\omega J_1(\lambda\omega a)]}. \tag{14}$$

The pinch ratio Θ is related to the ratio of the current and flux, and is a physically controllable quantity. For the Taylor state, $\Theta = \lambda a/2$.

The details of any relaxed state are determined by two physically meaningful parameters, the toroidal flux and the volts-seconds (Vs) of the discharge. The toroidal flux as defined earlier serves to determine the field amplitude, and the volts-seconds describes the helicity K of the relaxed state through the relation: $Vs = K/\Phi_z^2$. We therefore calculate the helicity integral (global helicity) from our solution for the $m = 0, k = 0$ state using

$$K = 4\pi^2 R\int_0^a (A_\theta B_\theta + A_z B_z) r\,dr . \tag{15}$$

For the minimum value of $\lambda a = 3.11$, the critical value of $Vs = 12.8$ R/a. For values of volts-seconds less than this critical value, a lower value of λa is obtained from solving the equation for K/Φ_z^2 so that the cylindically symmetric state is the relaxed state for minimum energy dissipation. For values of volts-seconds greater than the critical value the system relaxes to the helically distorted state with $\lambda a = 3.11$ which is obtained as a mixture of the $m = 0, k = 0$ and the $m = 1, k \neq 0$ states as in the case of Taylor's theory.

The values of both F and Θ at the boundary $r = a$ are evaluated, and F is plotted against pinch ratio Θ (Fig. 1). It is observed that F reverses at a value of $\Theta = 2.4$ ($\lambda a = 2.95$), whereas for the Taylor state the reversal is achieved at $\Theta = 1.2$. However, this field reversed state supports pressure gradient, in constrast to the Taylor state.

The pressure profile can be obtained from the relation $\mathbf{j} \times \mathbf{B} = \nabla p$. For the $m = 0, k = 0$ state, the only nonvanishing component of the pressure gradient exists in the radial direction. The pressure profile is shown in Fig. 2 for the $m = k = 0$ state with $\lambda a = 3.0$ which is the minimum energy dissipation, field-reversed state. Also, an increasing trend of plasma beta (ratio of volume averaged plasma pressure to the volume averaged magnetic pressure) with the pinch ratio has been observed.

In conclusion, the principle of minimum dissipation is utilized together with the constraints of constant magnetic helicity to determine the relaxed states of a

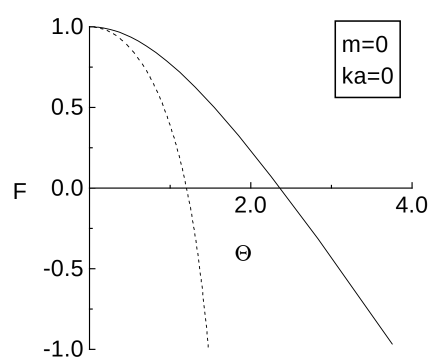

FIG. 1. The field reversal parameter F against the pinch parameter Θ, the field reversal occurring at $\Theta = 2.4$. The dotted curve represents the plot for the minimum energy state of Taylor.

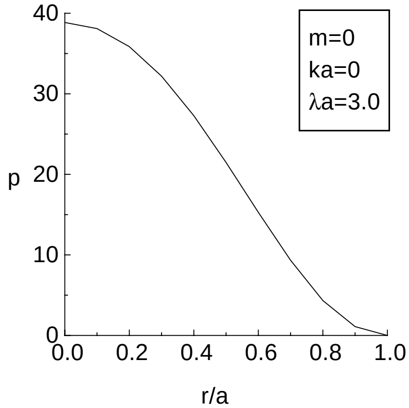

FIG. 2. The pressure profile p vs r for $\lambda a = 3.0$.

magnetoplasma. The variational principle leads to a remarkable Euler-Lagrange equation, and it is shown that this equation involving higher order curl operator can be solved in terms of an analytical continuation of CK functions in the complex domain with appropriate boundary conditions. This relaxed state obtained from single fluid MHD supports pressure gradient. A coupling between magnetic field and flow is not an essential criterion for having a nonzero pressure gradient. Further, it is shown that a non-force-free state with field reversal properties can exist.

However, we have considered only the Ohmic dissipation in our discussion. A more complete theory of relaxation should probably include other dissipation mechanisms, such as thermal transport, and such a theory may do better justice to the problem of plasma relaxation based on the principle of minimum dissipation.

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