Fractional Dynamics in Random Velocity Fields

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We develop an iterative approach, from which an *exact* systematic calculation of the moments of the probability density of test particles, under a generalized Fickian evolution and in the presence of a *stratified* random velocity field, can be calculated. The type of fractional diffusion-advection equation studied is useful for transport in anisotropic and heterogeneous media. Our method stresses the relevance of the coupling between anomalous diffusion and convection by stratified random velocity fields. [S0031-9007(98)07342-6]

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Sedimentary ground materials like rocks, soils, and wood products are notoriously anisotropic and heterogeneous. Transport and flow through such porous structures have important applications in science. Oil extraction from rock reservoirs is a clear example of them [1]. Moreover, anomalous diffusion has been recorded in experiments for these systems of environmental interest [2]. Thus, the interplay between anomalous diffusion and anisotropic transport seems to be of fundamental interest.

Numerous applications of the theory of anomalous diffusion arise in the context of solid state physics in composite materials. Different approximations have been introduced to describe transport in random media [3]. Among these approaches the effective medium approximation (introduced by Kirkpatrick [4], Orbach *et al.* [5], and Lax *et al.* [6]) has proved to be a very good tool to tackle the problem of anomalous diffusion in isotropic situations. Nevertheless, little effort has been directed to anisotropic random systems [7]. Such an extension is not trivial and leads to interesting predictions concerning the conductivity and its anisotropy in nonisotropically disordered systems [8]. In studying anomalous diffusion taking place in a flowing fluid, the continuous-time random walk (CTRW) theory [9] has shown to be a good approximation [10]. A remarkable point is that the CTRW theory can also be mapped to *fractional dynamics* [11]. This fact is very convenient for studying anomalous diffusion taking place in a *randomly* flowing fluid. Understanding the mechanism of anisotropic transport in heterogeneous media will have fundamental consequences for the future development of the theory of anomalous *pure diffusion*, as well as anomalous diffusion taking place in a *flowing fluid*. A unified framework to tackle this problem is the so-called fractional diffusion-advection dynamics.

Fractional partial differential equations contain *fractional* rather than integer-order derivatives [12]. The concept of fractional differentiation is an old subject [13]. Nevertheless, only recently it has been applied to provide a theoretical framework for studying transport in porous media [14] and to study *superslow* diffusion on fractal structures [15]. From the mathematical point of view, *time* fractional calculus has been applied in diffusionlike problems [16,17]. Fractional equations of motion arise also at *an equilibrium* phase transitions, or whenever a dynamical system is restricted to subsets of measure zero of its state space [18]. Recent publications [11,19] have reopened the discussion on the fractional partial differential equations as an alternative way to study anomalous *sub*diffusion [i.e., the mean square displacement of the test particle $\langle \mathbf{r}^2(t) \rangle$ grows more slowly than linearly with time], and to study anomalous superdiffusion to model turbulence in fluids [20] [i.e., $\langle \mathbf{r}^2(t) \rangle$ grows faster than linearly with time]. On the other hand, direct verifications performed on cethyl trimethyl ammonium bromide molecules "*living polymers*" have shown Lévy flights *super*diffusion [21], which can be modeled through fractional diffusion equations [11].

Diffusion interacts with other transport processes, for example diffusing tracers in confined gradient flows is a well understood model (Taylor dispersion). Diffusion tracers on stratified flows is also an interesting problem that has been traditionally studied in the context of ground water transport in geological aquifers [22,23]. The pioneer theoretical model involving a stratified random velocity field was constructed by Matheron and de Marsily [22], who showed the unusual behavior of the longitudinal dispersion $\langle x^2(t) \rangle \sim t^{3/2}$. Nevertheless, up to now all these analyses were done only under the Fickian diffusion approximation [24,25]. Here we extend this treatment in several respects to gain a deeper insight into this problem. With this aim we consider a general fractional diffusion equation in the presence of a stratified quenched random velocity field (with a *short* or *long*-range velocity correlation). To our knowledge, such an analysis has not yet been presented despite its interest for some authors [26]. We will be able to consider subdiffusion or superdiffusion and to study its interplay with the random velocity field.

The purpose of the present communication is to present a systematic method that enables one to obtain exact

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analytical solutions for the moments of the density of test particles. We emphasize that in the present Letter we give an exact evaluation of the second moment for *any* correlation of the velocity field. The convergence of our perturbation method rests on the stratified character of the velocity field, and on the isotropy and sufficiently fast decay—in Fourier space—of the non-Gaussian diffusive propagator. This convergence is guaranteed for a broad class of "long-tail" anomalous diffusive motions [those satisfying Eq. (9) below]. In this Letter we present two examples where the velocity correlation has an exponential decay, and one application with a power-law decay to emphasize the relevance of our theory.

In the particular case of a superdiffusive walker in the presence of an *isotropic* random (white) velocity field, a fractional formulation has been proposed in the form of a Laplacian-like operator [27,28]. There, perturbative approximations have been performed to study $\langle \mathbf{r}^2(t)\rangle$ by means of a field-theoretic renormalization group analysis. However, in our *stratified* random velocity model, the iterative approach itself is made in terms of the non-Gaussian propagator in the absence of flow. This fact allows us to get an exact evaluation of the second moment of the tracer for any correlation of the stratified velocity field.

Anomalous diffusion is a very general concept and has been analyzed from a variety of standpoints [5,6,29]. Here we will focus on the so-called fractional diffusion models, where the advection-diffusion equations take the form [10]

$$\frac{\partial \varrho(\mathbf{r},t)}{\partial t} + A\nabla\cdot\left[\mathbf{V}(\mathbf{r})\,\frac{\partial^{1-\gamma}\varrho(\mathbf{r},t)}{\partial t^{1-\gamma}}\right] = D\nabla^2 \times \frac{\partial^{1-\gamma}\varrho(\mathbf{r},t)}{\partial t^{1-\gamma}}, \tag{1a}$$

$$\frac{\partial \varrho(\mathbf{r},t)}{\partial t} + \nabla\cdot[\mathbf{V}(\mathbf{r})\varrho(\mathbf{r},t)] = D\nabla^{2\beta}\varrho(\mathbf{r},t), \tag{1b}$$

whose non-Gaussian propagators [see Eqs. (3) below] have been shown to be well behaved. Here $0 < \gamma < 1$, $\frac{1}{2} < \beta < 1$, and A is a parameter that might be interpreted as a generalized mobility [in which case $\mathbf{V}(\mathbf{r})$ would stand for a force field], i.e., this macroscopic parameter sets the right dimensions in the advection term [10]. In (1a) and (1b) the operators $\nabla^{2\beta}$ and $\partial^\alpha/\partial t^\alpha$ are defined through inverse Fourier and Laplace transforms, respectively [11]:

$$\frac{\partial^\alpha \varrho(\mathbf{r},t)}{\partial t^\alpha} = \mathcal{L}^{-1}\{u^\alpha \varrho(\mathbf{r},u)\}, \qquad \alpha < 1,$$

and

$$\nabla^{2\beta}\varrho(\mathbf{r},t) = \mathcal{F}^{-1}\{-|\mathbf{k}|^{2\beta}\varrho(\mathbf{k},t)\}.$$

It has been seen that (1a) corresponds to fractal time diffusion and (1b) to Lévy flights [10]. In our further developments we work in the Fourier-Laplace domain, where (1a) and (1b) turn into

$$(u + D|\mathbf{k}|^2u^{1-\gamma})\varrho(\mathbf{k},u) = 1 - \frac{iAu^{1-\gamma}}{(2\pi)^d}\,\mathbf{k} \cdot \int d\mathbf{k}'\,\mathbf{V}(\mathbf{k}-\mathbf{k}')\varrho(\mathbf{k}',u), \tag{2a}$$

$$(u + D|\mathbf{k}|^{2\beta})\varrho(\mathbf{k},u) = 1 - \frac{i}{(2\pi)^d}\,\mathbf{k} \cdot \int d\mathbf{k}'\,\mathbf{V}(\mathbf{k}-\mathbf{k}')\varrho(\mathbf{k}',u). \tag{2b}$$

By defining

$$\Sigma(\mathbf{k},u) = \frac{u^{\gamma-1}}{u^\gamma + D|\mathbf{k}|^2}, \qquad \Delta(\mathbf{k},u) = \frac{A}{u^\gamma + D|\mathbf{k}|^2} \tag{3a}$$

or

$$\Sigma(\mathbf{k},u) = \frac{1}{u + D|\mathbf{k}|^{2\beta}}, \qquad \Delta(\mathbf{k},u) = \frac{1}{u + D|\mathbf{k}|^{2\beta}}, \tag{3b}$$

Eqs. (2a) and (2b) can be recast into the unitary form

$$\varrho(\mathbf{k},u) = \Sigma(\mathbf{k},u) - \frac{i}{(2\pi)^d} \times \int d\mathbf{k}'\,G(\mathbf{k},\mathbf{k}-\mathbf{k}',u)\varrho(\mathbf{k}',u), \tag{4}$$

where $G(\mathbf{k},\mathbf{k}',u) = \Delta(\mathbf{k},u)\mathbf{k}\cdot\mathbf{V}(\mathbf{k}')$. This expression is our starting point to compute the moments of $\varrho(\mathbf{r},t)$; note that $\Sigma(\mathbf{k},u)$ is the anomalous propagator. A similar approach has been used in [10,30] to study anomalous tracers in linear shear flows and in a Poiseuille flow.

We will focus on a *stratified* random medium, i.e., the velocity field is everywhere directed along the same direction, say axis x, and it varies only along the transversal direction, i.e., axis y: $\mathbf{V}(\mathbf{r}) = [V(y), 0]$. We imagine that we release a tracer at a point in our environment and we study what its dispersion is if we consider all the statistical realizations of the disorder:

$$\langle V(y)\rangle = 0,$$

$$\langle V(y')\cdots V(y^{(n)})\rangle = \sigma_n f_n(y'-y'')\cdots f_n(y^{(n-1)}-y^{(n)}),$$

$f_n(y)$ characterizing the stationary n-point moment of the field.

Short or long range correlations can be handled by using convenient $f_n(y)$. In Fourier space:

$$\langle V(\mathbf{k}')\cdots V(\mathbf{k}^{(n)})\rangle = \sigma_n(2\pi)^{n+1}\widetilde{f_n}(-k_y'')\cdots\widetilde{f_n}(-k_y^{(n)}) \times \delta(k_x')\cdots\delta(k_x^{(n)}) \times \delta(k_y' + \cdots + k_y^{(n)}). \tag{5}$$

For each configuration of the velocity field, we are interested in two longitudinal quantities: the mean position and the spread of the tracer, so we need

$$\overline{x} = i \frac{\partial \varrho(\mathbf{k},t)}{\partial k_x}\bigg|_{\mathbf{k}=0} \qquad \text{and} \qquad \overline{x^2} = -\frac{\partial^2 \varrho(\mathbf{k},t)}{\partial k_x^2}\bigg|_{\mathbf{k}=0}.$$

To compute $\overline{x}$ and $\overline{x^2}$ we first iterate indefinitely (4) to get

$$\varrho(\mathbf{k},u) = \Sigma(\mathbf{k},u) + \sum_{n=1}^{\infty}\left(\frac{-i}{(2\pi)^d}\right)^n \int d\mathbf{k}' \cdots \int d\mathbf{k}^{(n)}\, G(\mathbf{k},\mathbf{k}-\mathbf{k}',u) \times G(\mathbf{k}',\mathbf{k}'-\mathbf{k}'',u)\cdots G(\mathbf{k}^{(n-1)},\mathbf{k}^{(n-1)}-\mathbf{k}^{(n)},u)\Sigma(\mathbf{k}^{(n)},u), \tag{6}$$

we differentiate (6) with respect to k_x and substitute $\mathbf{k}=0$. This results in

$$\overline{x} = i\Delta(\mathbf{k}=0,u)\sum_{n=1}^{\infty}\left(\frac{-i}{(2\pi)^d}\right)^n \int d\mathbf{k}' \cdots \int d\mathbf{k}^{(n)}\, k_x'\Delta(\mathbf{k}',u)\cdots k_x^{(n-1)}\Delta(\mathbf{k}^{(n-1)},u)\Sigma(\mathbf{k}^{(n)},u) \times V(-\mathbf{k}')V(\mathbf{k}'-\mathbf{k}'')\cdots V(\mathbf{k}^{(n-1)}-\mathbf{k}^{(n)}), \tag{7}$$

and in short notation $[\Delta(\mathbf{k}) \equiv \mathcal{F}\mathcal{L}[\Delta(\mathbf{x},t)]$, etc.].

$$\overline{x^2} = -\frac{\partial^2\Sigma}{\partial k_x^2}\bigg|_{\mathbf{k}=0} - \frac{i}{\pi}\Delta(\mathbf{k}=0)\int d\mathbf{k}'\,\frac{\partial V(-\mathbf{k}')}{\partial k_x'}\Sigma(\mathbf{k}') + 2\Delta(\mathbf{k}=0)\sum_{n=2}^{\infty}\left(\frac{-i}{2\pi}\right)^n\int d\mathbf{k}'\cdots\int d\mathbf{k}^{(n)}\,\frac{\partial V(-\mathbf{k}')}{\partial k_x'} \times G(\mathbf{k}',\mathbf{k}'-\mathbf{k}'')\cdots G(\mathbf{k}^{(n-1)},\mathbf{k}^{(n-1)}-\mathbf{k}^{(n)})\Sigma(\mathbf{k}^{(n)}). \tag{8}$$

Equations (6)–(8) must be seen as formal expressions since $V(\mathbf{r})$ is in general a stochastic nondifferentiable function. Only after taking disorder averages we expect to obtain finite computable quantities.

Eventually, we want to average over all the configurations of the velocity field and use (5), whence we need to remove every differentiation over the velocities in (8). To this aim we successively integrate (8) by parts observing that

$$[\Sigma(\mathbf{k},u)]_{\mathbf{k}=\pm\infty} = 0 \qquad \text{and} \qquad [k_x\Delta(\mathbf{k},u)]_{\mathbf{k}=\pm\infty} = 0 \tag{9}$$

for the cases that we consider in (3).

We remark that taking the average over the field, with the help of (5), $\langle\overline{x}\rangle$, and many terms in $\langle\overline{x^2}\rangle$ vanish because of the stratification and the symmetry of $\Sigma(\mathbf{k},u)$ and $\Delta(\mathbf{k},u)$.

The resulting *exact* formula for the tracer dispersion is

$$\langle\overline{x^2}\rangle(u) = -\frac{\partial^2\Sigma(\mathbf{k},u)}{\partial k_x^2} + \frac{\sigma_2}{\pi}\Delta(\mathbf{k}=0,u)\Sigma(\mathbf{k}=0,u) \times \int_{-\infty}^{\infty} dk_y\,\Delta(k_x=0,k_y,u)\tilde{f}_2(-k_y). \tag{10}$$

Similarly, from (7) we compute the average over the velocity field of the quantity $\overline{x}^2$

$$\langle\overline{x}^2\rangle(t) = \frac{\sigma_2}{2\pi}\int_{-\infty}^{\infty} dk_y\,\tilde{f}_2(-k_y) \times [\mathcal{L}^{-1}\{\Delta(\mathbf{k}=0,u)\Sigma(k_x=0,k_y,u)\}]^2. \tag{11}$$

The results (10) and (11) are exact and compare beautifully to the results of Matheron and de Marsily [22] and Bouchaud *et al.* [25] for the case of Fickian diffusion and white correlation. Indeed, let us take (3b) for $\beta = 1$ (Fickian diffusion), $f_2(y) = \delta(y)$, and let us calculate $\langle\overline{x^2}\rangle$ and $\langle\overline{x}^2\rangle$. The results are

$$\langle\overline{x^2}\rangle = 2Dt + \frac{4\sigma_2}{3\sqrt{\pi D}}t^{3/2} \qquad \text{and} \qquad \langle\overline{x}^2\rangle = (\sqrt{2}-1)\frac{4\sigma_2}{3\sqrt{\pi D}}t^{3/2} \tag{12}$$

in accordance with [22,25]. Moreover, our approach is more general since it can also describe nonwhite velocity correlations and non-Fickian diffusion in an exact formulation.

Now we present the results for non-Fickian diffusion in the presence of a short-range velocity field in two different cases: fractal time diffusion, given by (3a), and Lévy flights, as in (3b). To model short-range correlations we take $\tilde{f}_2(k) = \exp(-\mathrm{L}^2k^2)$. In the first case, substitution of (3a) in (10) and (11) leads, through the suitable inverse Laplace formulas and for times longer than $(\mathrm{L}^2/D)^{1/\gamma}$, to the asymptotic results

$$\langle\overline{x^2}\rangle \simeq \frac{2D}{\Gamma(1+\gamma)}t^{\gamma} + \frac{A^2\sigma_2}{\sqrt{D}\,\Gamma(1+3\gamma/2)}t^{3\gamma/2} \tag{13}$$

and

$$\langle\overline{x}^2\rangle \simeq \frac{A^2\sigma_2}{\pi\sqrt{D}}t^{3\gamma/2}\int_0^{\infty}E^2_{\gamma,1+\gamma}(-\varsigma^2)d\varsigma \propto \frac{A^2\sigma_2}{\sqrt{D}}t^{3\gamma/2}, \tag{14}$$

$E_{\gamma,1+\gamma}(x)$ being the generalized Mittag-Leffler function. Since $\gamma \in [0,1]$ the resulting dispersion is slower than for Fickian diffusion in the same situation, though it becomes superdiffusive for $\frac{2}{3} < \gamma < 1$.

In the case of Lévy flights, some additional caution is necessary since the second moment is known to diverge. This divergence also occurs in our case when one substitutes (3b) into the first term in (10), which is the term arising in diffusion in a resting background, but the correction [second summand in (10)] remains finite. Here, we follow the scaling arguments presented in [27] to avoid the divergence by restricting our integration to a box of width L, where L scales in a prescribed way with time. We thus consider the quantity $\overline{x^2}_L = \int_0^L x^2 \rho(x,t)\,dx$ and let L grow with time as $L = \mathcal{A}(Dt)^{1/2\beta}$. The first term in (6) is thus seen to contribute finitely to $\langle \overline{x^2} \rangle_L$ and the second term yields its asymptotic finite contribution as $L \longrightarrow \infty$ provided its typical scaling for x grows slower than L for growing times. Such a scaling for L also ensures that at any time we are including the same number of particles in our box. So, asymptotically

$$\langle \overline{x^2} \rangle_L \simeq g_\beta(\mathcal{A})\,(Dt)^{1/\beta} + \frac{1}{\pi}\,\frac{2\beta\Gamma(1/2\beta)}{(4\beta - 1)(2\beta - 1)} \times \frac{\sigma_2}{D^{1/2\beta}}\,t^{2-1/2\beta}, \tag{15}$$

where $g_\beta(\mathcal{A})$ is a certain function of the parameter $\mathcal{A}$, and the parameter β must be further restricted to $\frac{1}{2} < \beta < \frac{3}{4}$ in order to fulfill the conditions that we discussed for the scaling of L.

Thus, the correction due to the random velocity field is remarkably smaller than for Fickian diffusion [compare with (12)]. We see that the intrinsic superdiffusion of the Lévy flights, $\sim t^{1/\beta}$, dominates the correction due to the random velocity environment $\sim t^{2-1/2\beta}$.

On the other hand, when $\frac{1}{2} < \beta < 1$, the variance of the mean is finite and reads

$$\langle \overline{x}^2 \rangle \simeq \frac{\sigma_2}{\pi D^{1/2\beta}}\,t^{2-1/2\beta} \int_0^\infty \left[\frac{1 - \exp(-\varsigma^{2\beta})}{\varsigma^{2\beta}} \right]^2 d\varsigma \propto \frac{\sigma_2}{D^{1/2\beta}}\,t^{2-1/2\beta}. \tag{16}$$

We point out that when the velocity field is parallel to the stratification, the analysis of sample-to-sample fluctuations, and the influence of long-range velocity correlations can also be made *in an exact way*. For instance, consider a Brownian walker in a correlated long-range velocity field such that $f_2(y) = l^{-1}/(1 + |y|/l)^\mu$ with $0 < \mu < 1$, then Eq. (10) yields for the leading term of the tracer dispersion, as $t \longrightarrow \infty$,

$$\langle \overline{x^2} \rangle \simeq \frac{2\sigma_2}{D^{\mu/2} l^{1-\mu}}\,\frac{\Gamma(1 - \mu)}{\Gamma(3 - \frac{\mu}{2})}\,t^{2-\mu/2}. \tag{17}$$

Thus, the longitudinal dispersion of a Brownian particle in a *stratified* power-law correlated field will strongly depend on the long tail of the long-range correlation. For a long-range correlation with $\mu = 1$ the asymptotic behavior goes like $\sim t^{3/2}\ln(t)$; only for $\mu > 1$ we obtain a longitudinal dispersion $\sim t^{3/2}$ as in [22]. The interesting interplay between anomalous diffusion (non-Fickian case) and long-range velocity correlation will be reported elsewhere.

As a final remark we comment that the isotropic case, when the velocity field is *nonstratified*, can also be handled *in an exact way* if the velocity correlation is white.

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[1] *Flow Through Porous Media*, edited by R. M. de Wiest (Academic Press, New York, 1969).
[2] B. Berkowitz and H. Scher, Phys. Rev. E **57**, 5858 (1998).
[3] J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[4] S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).
[5] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981).
[6] T. Odagaki and M. Lax, Phys. Rev. B **24**, 5284 (1981).
[7] E. R. Reyes, M. O. Cáceres, and P. A. Pury, Physica (Amsterdam) **258A**, 1 (1998).
[8] J. Bernasconi, Phys. Rev. B **9**, 4575 (1974).
[9] H. Scher and M. Lax, Phys. Rev. B **7**, 4491 (1973).
[10] A. Compte, Phys. Rev. E **55**, 6821 (1997).
[11] A. Compte, Phys. Rev. E **53**, 4191 (1996).
[12] K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974).
[13] A. Gemant, Physics (N.Y.) **7**, 311 (1936); G. W. Scott Blair, Proc. R. Soc. London **189**, 69 (1947).
[14] B. O'Shaughnessy and I. Procaccia, Phys. Rev. Lett. **54**, 455 (1985).
[15] R. R. Nigmatullin, Phys. Status. Solidi (b) **133**, 425 (1986).
[16] W. Wyss, J. Math Phys. **27**, 2782 (1986).
[17] G. Jumarie, J. Math Phys. **33**, 3536 (1992).
[18] R. Hilfer, Phys. Rev. E **48**, 2466 (1993).
[19] M. Giona and H. E. Roman, J. Phys. A **25**, 2093 (1992); J. Phys. A **25**, 2107 (1992); Physica (Amsterdam) **185A**, 87 (1992); H. E. Roman and P. A. Alemany, J. Phys. A **27**, 3407 (1994); R. Hilfer and L. Anton, Phys. Rev. E **51**, R848 (1995).
[20] M. F. Shlesinger, B. J. West, and J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987).
[21] A. Ott, J.-P. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990).
[22] G. Matheron and G. de Marsily, Water Resour. Res. **16**, 901 (1980).
[23] G. Dagan, Annu. Rev. Fluid Mech. **19**, 183 (1987).
[24] M. Avellaneda and A. J. Majda, J. Stat. Phys. **69**, 689 (1992); P. LeDoussal, J. Stat. Phys. **69**, 917 (1992); B. Gaveau and L. S. Schulman, J. Stat. Phys. **66**, 375 (1992); D. ben-Avraham, F. Leyvraz, and S. Redner, Phys. Rev. A **45**, 2315 (1992).
[25] J.-P. Bouchaud, A. Georges, J. Koplik, A. Provata, and S. Redner, Phys. Rev. Lett. **64**, 2503 (1990).
[26] R. M. Mazo, Acta Phys. Pol. B **29**, 1539 (1998).
[27] H. C. Fogedby, Phys. Rev. Lett. **73**, 2517 (1994).
[28] J. Honkonen, Phys. Rev. E **53**, 327 (1996).
[29] J. W. Haus and K. W. Kehr, Phys. Rep. **150**, 263 (1987); S. Havlin and D. ben-Avraham, Adv. Phys. **36**, 695 (1987); M. O. Caceres, H. Matsuda, T. Odagaki, D. P. Prato, and W. Lamberti, Phys. Rev. B **56**, 5897 (1997).
[30] A. Compte, R. Metzler, and J. Camacho, Phys. Rev. E **56**, 1445 (1997).