Noise Induced Pattern Transition and Spatiotemporal Stochastic Resonance

Hou Zhonghuai, Yang Lingfa, Xiaobin Zuo, and Xin Houwen*
Department of Chemical Physics, University of Science and Technology of China, Hefei, Anhui, 230026, People's Republic of China
(Received 16 January 1998)

The influence of parametric noise on the spatiotemporal dynamics of a two-variable reaction-diffusion model, describing pattern formation in excitable media, is investigated. When a control parameter is perturbed by stochastic force with suitable intensity, noise induced pattern transition between "single" spiral wave and "double" spiral waves, which does not occur in the absence of noise, is observed. If this parameter also varies periodically in time around the bifurcation point, we find that the output signal-to-noise ratios show maxima with the variation of noise intensity, indicating the occurrence of spatiotemporal stochastic resonance. [S0031-9007(98)07070-7]

PACS numbers: 05.40.+j, 05.45.+b, 47.54.+r, 87.10.+e

The influence of noise on nonlinear systems is the subject of intense experimental and theoretical investigations. The most well-known phenomena is noise induced transition [1] and stochastic resonance (SR) [2], both showing the possibility to transform noise into *order*.

For zero-dimensional systems coupled with noise in multiplicative ways, the bifurcation point is triggered by the noise and the bifurcation character is quite different from the deterministic system. Transitions into new stationary states may occur, which is purely induced by noise, henceforth called noise induced transition [3,4]. Recent research has paid much attention to spatial extended systems, such as reaction-diffusion system [5,6], global coupled systems [7], etc. It is found that noise can lead to new spatiotemporal structures which does not exist in the deterministic system.

The phenomenon of SR has gained intense attention due to its great number of applications in different fields of science [2]. This behavior is characterized by the possibility of enhancement of output signal-to-noise ratio (SNR) with the addition of an optimized amount of noise, which is somewhat counterintuitive. After originally being proposed by Benzi *et al.* to account for the Earth's ice ages [8], theories of SR have been developed for quite a wide variety of systems such as the following: monostable [9], multistable [10], excitable [11–13], neuronal network [14], and even threshold-free systems [15,16]. In addition, the input signal can be periodic or even chaotic [11,12,17], the noise can be Gaussian white noise or colored [18], which can be coupled with the system in additive or multiplicative [19] ways. If the nonlinear kinetics of the system satisfy particular conditions, SNR can show multimaxima, indicating the occurrence of stochastic multiresonance [20]. Recently, some authors have studied the influence of additive noise on spatial extended systems, e.g., spatial coupled model in excitable media [21], reaction-diffusion model [22], and coupled map lattice [23], when the control parameter of which varies periodically in time around the bifurcation point. It is found that the coherence of noise with the system also occurs, which is henceforth called spatiotemporal stochastic resonance (STSR).

In the present work, we study a two-variable reaction-diffusion model, describing pattern formation in excitable media, with the influence of parametric noise (therefore the noise is coupled with the system in a multiplicative way). The model is written as

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon} f(u, v) + D_u \nabla^2 u , \quad (1)$$

$$\frac{\partial v}{\partial t} = g(u, v) + D_v \nabla^2 v , \quad (2)$$

where $\epsilon \ll 1$, and $f(u,v)$ and $g(u,v)$ describe the nonlinear kinetics of specific systems. Most generally, the u nullcline $f(u,v) = 0$ is an N-shaped function in u-v plane and the v nullcline $g(u,v) = 0$ is monotone, intersecting the u nullcline at only one point (u_0, v_0), which is an excitable stable fixed point [24]. Choosing suitable $f(u,v)$ and $g(u,v)$, this model can describe the spatiotemporal dynamics of the Brillouin zone reaction [25], neural networks [13], and CO oxidation on catalytic surface [26,27], etc. Here we choose $f(u,v) = u(1-u)[u - (v+b)/a]$, $g(u,v) = u - v$, and $D_v = 0$. The simulation was carried out on a two-dimensional square lattice under a no-flux boundary condition through a fast algorithm proposed by Barkley [24]. The initial condition is illustrated in Fig. 1(a) or Fig. 3(a), which acts as a "seed" to form spiral waves. If $a < 1$, the model is excitable, and with increasing b, the system shows turbulence, spiral wave, flat wave front, even no wave. Notice for the no-flux boundary condition, the flat wave front vanishes when it moves out of the lattice, and the system tends to the absorbing state $u_0 = v_0 = 0$.

Considering now the parameter b is perturbed by noise, i.e., $b = b_0 + \xi(t)$, where $\xi(t)$ denotes Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t')$, here D is the noise intensity. In real systems, e.g., the CO oxidation system [26], this noise may result from thermal

 0031-9007/98/81(14)/2854(4)$15.00

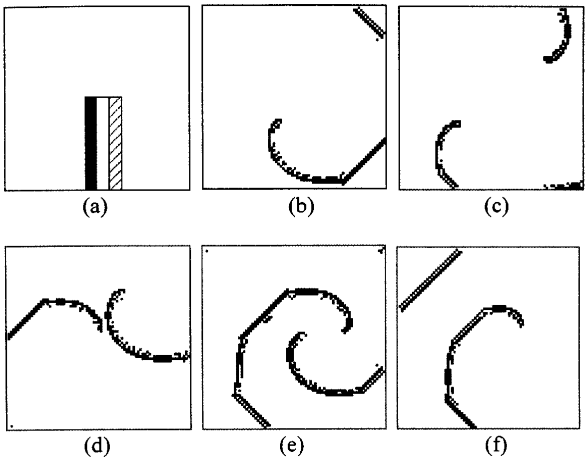

FIG. 1. Transition from SSW to DSW and then back to SSW. (a) Illustration of the initial condition: in the dark region, $u(i,j) = 0.7$, $v(i,j) = 0$; in the line-filled region, $u(i,j) = 0$, $v(i,j) = 0.7$; in the middle empty region, $u(i,j) = 0.7$, $v(i,j) = 0.5$, and here $u(i,j)/v(i,j)$ denotes the value of u/v on the lattice site (i,j). (b)–(f) Typical patterns. Notice the time interval between patterns are not equal.

fluctuation or fluctuation of CO partial pressure. Here we choose $a = 0.3$, $b_0 = 0.016$ to investigate the influence of parametric noise on the observed spatiotemporal dynamics. The initial condition is illustrated in Fig. 1(a). For $D = 0$, the asymptotic pattern is spiral wave with one branch, which we call "single" spiral wave (SSW); see Fig. 1(b). If the noise intensity is not zero but small, e.g., $D = 1 \times 10^{-4}$, the SSW may lose stability after a period of relaxation time and the system reduces to the absorbing state, which is also the case for large noise level ($D > 3 \times 10^{-3}$). For noise with intermediate intensity, however, one observes transitions into "double" spiral wave (DSW), which have two spiral branches rotating around each other, as it is shown in Fig. 1(e). At an optimized noise level, $D \simeq 8 \times 10^{-4}$, this transition behavior occurs the most frequently. The variation of u_{tot} (obtained by adding up all the value of u on the lattice sites) with time, for different noise level, is shown in Figs. 2(a)–2(d). Figure 2(e) presents the probability density distribution for $D = 0$ and $D = 8 \times 10^{-4}$. The latter one shows apparently the characteristic of noise induced bistability.

One notes that this kind of bistability arises via noise induced wave splitting, as it is shown in Fig. 1. For $D = 0$, the motion of the spiral core is deterministic and the SSW is rather stable. When noise is presented but very small, the trajectory of the spiral core is randomly perturbed and SSW would lose stability when the core moves out of the lattice. A larger noise, however, may lead to wave splitting, i.e., the SSW is broken into segments which rotate around their ends to form new spiral wave branches [Fig. 1(c)]. These new branches interact with the original ones, and DSW are formed after some time. The DSW can also jump back to the SSW if the open end of one spiral branch moves out of the lattice. If the noise intensity is too large, the SSW would lose stability before wave splitting occurs such that bistability disappears. It is found that the relative stability of SSW and DSW is also influenced by the lattice size, but the qualitative phenomena observed are similar.

For some other choices of parameters and initial conditions, more interesting phenomena are observed. An example is presented in Figs. 3 and 4. For $b_0 = 0.015$, $D = 0$, and choosing the initial condition illustrated in Fig. 3(a), the two branches of the spiral wave annihilate each other after a short time of evolution; accordingly the variation of u_{tot} with time is presented in Fig. 4(a). For $D = 1 \times 10^{-9}$, however, sustained wave propagation is found, and after a period of time, this SSW hops into DSW, which is rather stable [see Fig. 4(b)]. For $D = 1 \times 10^{-4}$, the transition between SSW and DSW is more frequent as it is shown in Fig. 4(c). Here a rather interesting phenomenon exhibits that the DSW can be clockwise (CW), e.g., Fig. 3(b), or anticlockwise (ACW), e.g., Fig. 3(d), showing the occurrence of noise induced "chirality" transition. This transition goes through a type of intermediate pattern in which CW- and ACW-wave branches are mixed, as it is shown in Fig. 3(c). It seems that CW-DSW and ACW-DSW are both asymptotic states of the system, while a specific initial condition only attains one type of them under deterministic evolution. We show that an introduction of suitable parametric noise can lead to transition between them, which seems to imply a possible important role of noise in chirality broken phenomena. For the control parameters and initial condition corresponding to Fig. 2(c), however, it is observed that the chirality transition can occur only when the SSW jumps into the DSW or vice versa, e.g., a CW-SSW can jump into an ACW-DSW, while direct transitions between CW-DSW and ACW-DSW were not observed.

From the preceding study, it is found that the parametric noise of b can lead to transition into new asymptotic patterns which may be of more interest, showing the constructive role of noise in shaping, and even controlling the observed spatiotemporal dynamics. Furthermore, there seems to exist an optimized noise level that supports the transition most which implies a possible resonance behavior between the noise and the system. One may expect that this kind of resonance behavior is relevant to stochastic resonance. Now we let the parameter b vary periodically around the bifurcation point in the presence of noise, i.e., $b = b_0 + A \sin \omega t + \xi(t)$. We choose $b_0 = 0.017$ and $A = 0.002$ ($\omega = \pi/50$) such that for $D = 0$, the combination of b_0 and $A \sin \omega t$ yet cannot "fire" sustained wave propagation (one notes that this is something like the threshold-crossing FHN model [13]). For these parameters, flat wave fronts prevail over spiral waves, and finally the wave fronts move out of the lattice. However, it is quite surprising that a rather small noise

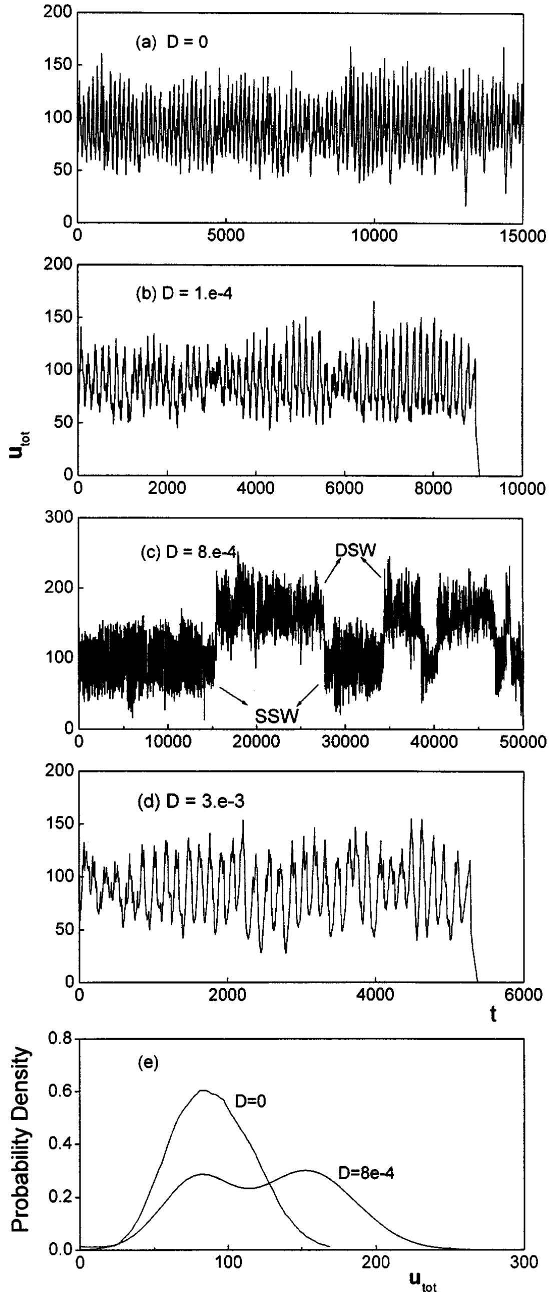

FIG. 2. (a)–(d) The variation of u_{tot} with time for different noise levels according to the initial condition illustrated in Fig. 1(a). (e) The probability density distribution for $D = 0$ and $D = 8 \times 10^{-4}$. The parameters are $a = 0.3$, $b_0 = 0.016$, $D_u = 0.2$, and lattice size $L = 80$.

level, e.g., $D = 1 \times 10^{-15}$, can support sustained wave traveling. Competition between spiral waves and flat wave fronts is subtly modulated by the signal and noise, which may lead to sustained oscillation of u_{tot} consisting of the frequency component of ω. Performing Fourier transform of the time series of u_{tot}, one finds a clear peak at the input frequency, where we can calculate SNR. The variation of SNR with noise intensity D is shown in Fig. 5, which undulates somewhat coherence behavior. For a very small

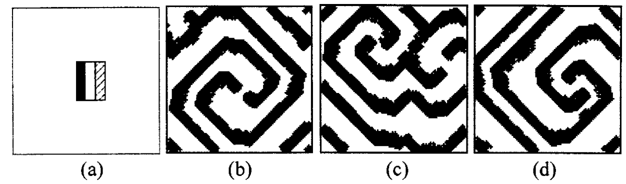

FIG. 3. Illustration of initial condition for Fig. 3 and typical patterns. (a) Initial condition: same as in Fig. 2(a) except that the places of the corresponding region are moved. (b) Example for clockwise DSW. (c) Intermediate pattern: mixed DSW. (d) Anticlockwise DSW.

noise level (e.g., $D \leq 1 \times 10^{-18}$), no sustained wave propagation can be fired, while for large noise intensity (e.g., $D \geq 1 \times 10^{-2}$), all periodic signals are submerged by noise.

In the present work, we have studied a two-variable reaction-diffusion model, describing pattern formation in excitable media, under the influence of parametric noise. Noise induced pattern transition between single spiral wave and double spiral waves is observed, which indicates that noise plays an important role in shaping the observed spatiotemporal dynamics. A rather interesting phenomena, noise induced chirality transition, occurs for

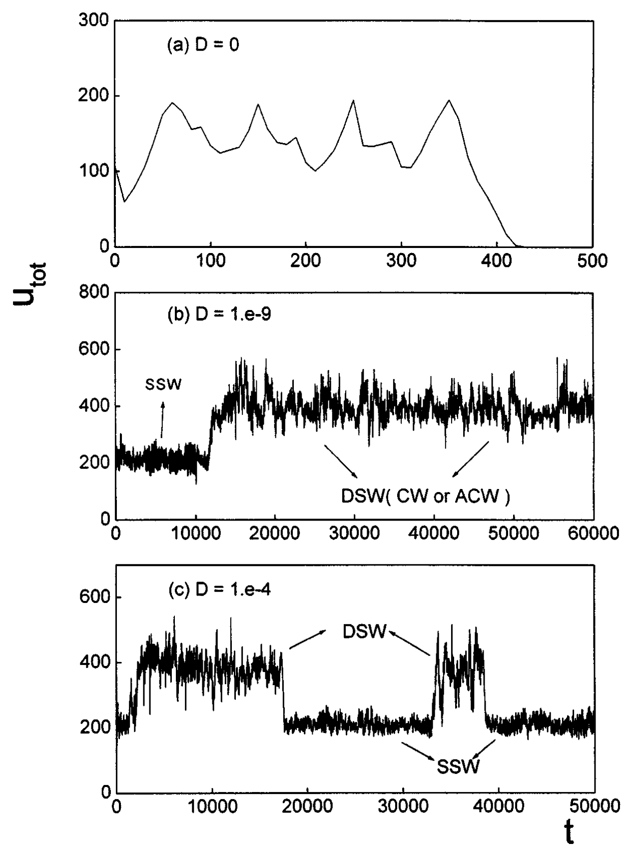

FIG. 4. The variation of u_{tot} with time for different noise levels according to the initial condition illustrated in Fig. 4(a). The parameters are $a = 0.3$, $b_0 = 0.015$, $D_u = 0.2$, and lattice size $L = 100$.

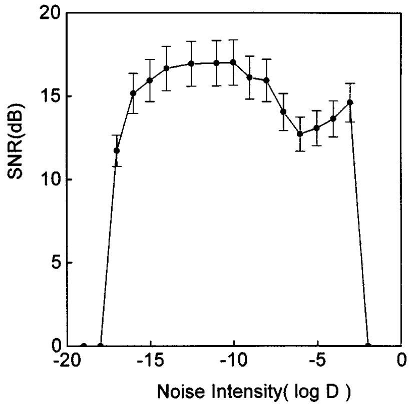

FIG. 5. Variation of signal-to-noise ratio (SNR) with the noise intensity D, calculated from the time series of $u_{\rm tot}$; the error bars are obtained by statistics on independent runs. The parameters are $a = 0.3$, $b_0 = 0.017$, $D_u = 0.2$, $A = 0.002$, and $\omega = \pi/50$.

some specific choices of parameters and initial condition, which may stimulate further study on the role of noise in chirality broken phenomena. In addition, it is found that an optimal noise level exists which supports the transition most. In another part of this paper, we show that this resonance behavior might be characteristic of spatiotemporal stochastic resonance: When the control parameter varies periodically in time around the bifurcation point, under the perturbation of noise, it is found that sustained wave propagation and oscillations of $u_{\rm tot}$ occur, and the output signal-to-noise ratio shows a maximum with the variation of noise intensity D. Since the model studied here is applicable for many important chemical or biological systems and parametric noise is quite universal resulting from fluctuations in temperature or partial pressure, the present work can help in understanding the observed dynamics of such systems and open up further perspectives about the role of multiplicative noise on pattern formation in nonlinear systems.

This work is supported by the National Science Foundation of China and the National Laboratory of Theoretical and Computational Chemistry of China.

*Author to whom correspondence should be addressed.

[1] W. Horsthemke and R. Lefever, *Noise-induced Transitions* (Springer-Verlag, Berlin, 1984).
[2] Kurt Wiesenfeld and Frank Moss, Nature (London) **373**, 33 (1995).
[3] Jennifer Foss, Frank Moss, and John Milton, Phys. Rev. E **55**, 4536 (1997).
[4] Fritz Gassmann, Phys. Rev. E **55**, 2215 (1997).
[5] Jorg Enderlein and Lothar Kuhnert, J. Phys. Chem. **100**, 19 642 (1996).
[6] Srinivas S. Yerrapragada, Jayanta K. Bandyopadhyay, V. K. Jayaraman, and B. D. Kulkarni, Phys. Rev. E **55**, 5248 (1997).
[7] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994); R. Muller, K. Lippert, A. Kuhnel, and U. Behn, Phys. Rev. E **56**, 2658 (1997); S. Mangioni, R. Deza, H. S. Wio, and R. Toral, Phys. Rev. Lett. **79**, 2389 (1997).
[8] R. Benzi, S. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981).
[9] J. M. G. Vilar and J. M. Rubi, Phys. Rev. Lett. **77**, 2863 (1996).
[10] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **62**, 349 (1989).
[11] J. J. Collins, Carson C. Chow, and Thomas T. Imhoff, Phys. Rev. E **52**, R3321 (1995).
[12] C. Eichwald and J. Walleczek, Phys. Rev. E **55**, R6315 (1997).
[13] Arkady S. Pikovsky and Jurgen Kurths, Phys. Rev. Lett. **78**, 775 (1997).
[14] Bruce J. Gluckman, Theoden I. Netoff, Emily J. Neel, Mark L. Spano, and Steven J. Schiff, Phys. Rev. Lett. **77**, 4098 (1996).
[15] P. Jung and K. Wiesenfeld, Nature (London) **385**, 291 (1997).
[16] Sergey M. Bezrukov and Igor Vodyanoy, Nature (London) **385**, 319 (1997).
[17] Rolando Castro and Tim Sauer, Phys. Rev. Lett. **79**, 1030 (1997).
[18] Claudia Berghaus, Angela Hilgers, and Jurgen Schnakenberg, Z. Phys. B **100**, 157 (1996).
[19] V. Berdichevsky and M. Gitterman, Europhys. Lett. **36**, 161 (1996).
[20] J. M. G. Vilar and J. M. Rubi, Phys. Rev. Lett. **78**, 2882 (1997).
[21] Peter Jung and Gottfried Mayer-Kress, Phys. Rev. Lett. **74**, 2130 (1995).
[22] J. M. G. Vilar and J. M. Rubi, Phys. Rev. Lett. **78**, 2886 (1997).
[23] Prashant M. Gade, Renuka Rai, and Harjinder Singh, Phys. Rev. E **56**, 2518 (1997).
[24] D. Barkley, Physica (Amsterdam) **49D**, 61 (1991).
[25] S. Grill, V. S. Zykov, and S. C. Muller, Phys. Rev. Lett. **75**, 3368 (1995).
[26] M. Bar, N. Gottschalk, M. Eiswirth, and G. Ertl, J. Chem. Phys. **100**, 1202 (1994).
[27] M. Bar, A. K. Bangia, I. G. Kevrekidis, G. Haas, H. H. Rotermund, and G. Ertl, J. Phys. Chem. **100**, 19 106 (1996).