Exact Solution of Vlasov Equations for Quasineutral Expansion of Plasma Bunch into Vacuum

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In the quasineutral approximation an exact self-similar solution of the three-dimensional Vlasov equations for electrons and ions with self-consistent electric field is obtained. This solution is valid for an arbitrary value of electron-to-ion mass ratio, arbitrary relation between electron and ion thermal energies, and for a large variety of electron velocity distribution functions and initial spatial plasma density distribution. The results obtained can be used as a basic model for studying the problem of expansion of an arbitrary two-component plasma. [S0031-9007(98)07204-4]

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Plasma expansion into vacuum has received significant attention from physicists over the past decades. Following the pioneer work by Gurevich *et al.* [1], most studies of the problem have been based on a model of semi-infinite plasma expansion [2–10]. Actually, this approach corresponds to the case of plasma flow generation by continuous sources of plasma present in a certain region of space. The specific feature of the solution obtained within this model is the occurrence of ion distribution function excessively enriched with energetic particles (comparing with electron distribution). It is caused by an unlimited reserve of energy and particles in the initial plasma.

Another regime is observed during the expansion of an initially confined plasma bunch. The process in this case is essentially a nonstationary one. It is accompanied by considerable cooling of electrons and therefore the study of this process requires quite another approach. The analytical investigations of this problem were carried out using only a phenomenological hydrodynamic model [11]. At the same time, the numerical simulation done within the kinetic description of particle motion has demonstrated [12,13] that the present computer capabilities, unfortunately, do not allow the study of a real three-dimensional process of plasma bunch expansion. Thus the situation in this field can be considered as quite unsatisfactory compared to the case of semi-infinite plasma expansion. However, things changed in 1998. The first analytical self-similar solution to the problem has been published [14]. The presented results correspond to a one-dimensional case. They have been obtained within a quasineutral approach, adiabatic description of electron motion in a charge separation electric field, and cold ion approximation. The same assumptions have allowed us to solve the three-dimensional problem [15]. It was shown recently [16] that the solution can be derived even if the adiabatic approximation is dropped out. It turns out that the approximation of cold ions can also be rejected. It is demonstrated below that the quasineutral approach is the only assumption that is necessary for finding an exact solution of three-dimensional Vlasov equations for both electrons and ions in a self-consistent electric field.

The physical model which has been used for deriving an exact solution assumes that in a space with dimensionality ν ($\nu = 1, 2, 3$) there is a bunch of collisionless plasma whose distribution functions of ions and electrons are $f_i(\mathbf{v}, \mathbf{r}, t)$ and $f_e(\mathbf{v}, \mathbf{r}, t)$, respectively. The evolution of both of these functions during plasma expansion is described by the Vlasov kinetic equations,

$$\frac{\partial f_i}{\partial t} + (\mathbf{v}, \nabla_{\mathbf{r}})f_i - \left(\frac{Ze}{M} \nabla_{\mathbf{r}} \varphi, \nabla_{\mathbf{v}}\right) f_i = 0, \tag{1}$$

$$\frac{\partial f_e}{\partial t} + (\mathbf{v}, \nabla_{\mathbf{r}})f_e + \left(\frac{e}{m} \nabla_{\mathbf{r}} \varphi, \nabla_{\mathbf{v}}\right) f_e = 0. \tag{2}$$

In Eqs. (1) and (2), Ze and M are the ion charge and mass, m and $-e$ are the electron mass and charge, and $\varphi(\mathbf{r}, t)$ is the potential of charge-separation electric field which is consistent with the Poisson equation:

$$\sum_{k=1}^{\nu} \frac{\partial^2 \varphi}{\partial x_k^2} = 4\pi e(n_e - Zn_i), \qquad n_{e,i}(\mathbf{r}, t) \equiv \int f_{e,i}(\mathbf{v}, \mathbf{r}, t)\, d^\nu \mathbf{v}, \tag{3}$$

where x_k are the components of the vector $\mathbf{r}$, and $n_i(\mathbf{r}, t)$ and $n_e(\mathbf{r}, t)$ are ion and electron densities, respectively.

An approximation of quasineutrality is a commonly accepted approach to analyzing the expansion of a sufficiently dense plasma. It implies that

$$n_e(\mathbf{r}, t) = Zn_i(\mathbf{r}, t). \tag{4}$$

On the one hand, the approximation of quasineutrality imposes a definite restriction on possible initial electron and ion distribution functions. They should be chosen in a way to prevent excitation of Langmuir oscillations in plasma and particle overtaking. On the other hand, the assumption (4) makes it possible to omit the Poisson equation and express the electric field through the electron and ion distribution functions:

$$e \frac{\partial \varphi}{\partial x_j} = \frac{mM}{M + Zm} \sum_{k=1}^{\nu} \frac{\frac{\partial}{\partial x_k} \int v_k v_j (Zf_i - f_e)\, d^\nu \mathbf{v}}{\int f_e(\mathbf{v}, \mathbf{r}, t)\, d^\nu \mathbf{v}}, \tag{5}$$

where v_k are components of the vector $\mathbf{v}$. It should be pointed out that, for the hydrodynamic description of plasma motion, the Eq. (5) simply corresponds to pressure balance.

Equations (1), (2), and (5) comprise a complete set. The main purpose of this Letter is to obtain an exact solution of it.

The problem of quasineutral expansion of a plasma bunch is solved by applying the method proposed in our previous paper [16]. A self-similar solution to (1), (2), and (5) can be presented in the following form:

$$f_e(\mathbf{v}, \mathbf{r}, t) = F(G_1^{(e)}, \ldots, G_\nu^{(e)}), \tag{6}$$

$$f_i(\mathbf{v}, \mathbf{r}, t) = \frac{1}{Z} F(G_1^{(i)}, \ldots, G_\nu^{(i)}) \prod_{k=1}^{\nu} \frac{V_k^{(e)}(0)}{V_k^{(i)}(0)}, \tag{7}$$

$$G_k^{(e,i)} = \left(\frac{x_k}{l_k(t)}\right)^2 + \left(\frac{v_k - u_k(x_k, t)}{V_k^{(e,i)}(t)}\right)^2,$$

where F is an arbitrary function; $l_k(t)$ are time-dependent scale lengths of plasma density defined via the second moments of the electron (or ion) distribution function,

$$l_k^2(t) \equiv \frac{\int\int x_k^2 f_{e,i}(\mathbf{v}, \mathbf{r}, t)\, d^\nu\mathbf{v}\, d^\nu\mathbf{r}}{\int\int f_{e,i}(\mathbf{v}, \mathbf{r}, t)\, d^\nu\mathbf{v}\, d^\nu\mathbf{r}}; \tag{8}$$

u_k are components of a vector of electron and ion average (hydrodynamic) velocities $\mathbf{u}(\mathbf{r}, t)$:

$$\mathbf{u}(\mathbf{r}, t) \equiv \frac{\int \mathbf{v} f_{i,e}(\mathbf{v}, \mathbf{r}, t)\, d^\nu\mathbf{v}}{\int f_{i,e}(\mathbf{v}, \mathbf{r}, t)\, d^\nu\mathbf{v}}; \tag{9}$$

$V_k^{(e,i)}(t)$ are electron and ion thermal speeds along the corresponding directions x_k. They are defined as

$$(V_k^{(e,i)})^2 \equiv \frac{\int\int (v_k - u_k)^2 f_{e,i}\; d^\nu\mathbf{v}\, d^\nu\mathbf{r}}{\int\int f_{e,i}(\mathbf{v}, \mathbf{r}, t)\, d^\nu\mathbf{v}\, d^\nu\mathbf{r}}. \tag{10}$$

The expressions (6) and (7) will be exact solutions to Eqs. (1), (2), and (5) if thermal and hydrodynamic velocities of ion and electron motions satisfy the following relationships:

$$V_k^{(e,i)}(t) = V_k^{(e,i)}(0) l_k(0)/l_k(t), \tag{11}$$

$$u_k(x_k, t) = x_k \dot{l}_k(t)/l_k(t), \qquad \dot{l}_k \equiv dl_k/dt, \tag{12}$$

and the universal law of quasineutral expansion into vacuum of plasma bunch is valid for the dependencies $l_k(t)$:

$$l_k^2(t) = [l_k(0) + \dot{l}_k(0)t]^2 + c_k^2 t^2. \tag{13}$$

Here the ion sonic velocity has been introduced as a constant vector with the components c_k. They are determined by initial values of electron and ion thermal velocities,

$$c_k^2 = \frac{Zm[V_k^{(e)}(0)]^2 + M[V_k^{(i)}(0)]^2}{M + Zm}. \tag{14}$$

The presented solution allows one to calculate the spatial distribution of the electric field potential,

$$\varphi(\mathbf{r}, t) = \frac{m}{2e} \sum_{k=1}^{\nu} \frac{(V_k^{(i)})^2 - (V_k^{(e)})^2}{(1 + Zm/M) l_k^2(t)}\, x_k^2. \tag{15}$$

Evidently, Eq. (15) is violated at bunch periphery where the quasineutral approximation is not fulfilled. (A more detailed discussion of the subject is presented below.)

The exact solution obtained in the kinetic model for both types of particles coincides with the one found using a hydrodynamic equation for ion motion [16] if the initial values of ion thermal velocities $V_k^{(i)}(0)$ are supposed to be equal to zero. The adiabatic approximation presented in [14,15] can be obtained from the above solution in the case when the ions are cold and the electron-to-ion mass ratio is negligibly small, so that

$$Zm/M \ll 1, \qquad V_k^{(i)}(0) \ll \sqrt{Zm/M}\, V_k^{(e)}(0). \tag{16}$$

It should be noted that, even if the inequalities (16) are valid, the solution derived within the adiabatic approach describes only the process of quasineutral plasma expansion during a limited time interval when the characteristic electron velocities remain higher than the velocities of the accelerated ions: $u_k \ll V_k^{(e)}$.

It seems appropriate to start the discussion of the solution presented above with an analysis of its applicability to the description of a real process of a plasma bunch expansion into vacuum. Let us first consider the validity of the quasineutrality approximation. The obtained solution corresponds to the presence of space charge, whose density is homogeneous and proportional to the absolute value of the Laplacian of the electric potential. Therefore, the quasineutral approximation is valid in that part of the plasma where the frequency of Langmuir oscillation ω_p is sufficiently high:

$$\omega_p^2 \gg \sum_{k=1}^{\nu} \frac{(V_k^{(e)})^2 - (V_k^{(i)})^2}{(1 + Zm/M) l_k^2}. \tag{17}$$

It is remarkable that, according to the results obtained, the right part of the inequality (17) decreases faster than the left, since the electrons cool as the plasma expands. Consequently, if the applicability condition for the quasineutral approximation holds in the bulk of the plasma at $t = 0$, it will hold there at all times. This fact suggests that the results obtained give a correct description of the real process of plasma expansion.

According to Eqs. (6) and (7), the found solution allows us to investigate only the expansion of a plasma with similar electron and ion distribution functions. Moreover, self-similarity of the solution obtained restricts the possible initial spatial distributions of plasma density for a given distribution over velocities. In particular, the cases in which collisionless shock formation or excitation of Langmuir oscillations in plasma are expected cannot be described by the type of solutions found in this paper. Nevertheless, due to the fact that the solution obtained contains such arbitrary independent parameters as the initial scale lengths of plasma density and the average velocities of electron and ion thermal motion, it is possible to apply Eqs. (6), (7), (11), (12), and (14) to many cases of plasma expansion. For example, they describe the case of Maxwellian distribution function for both types of particles with an arbitrary relationship between the ion and electron temperatures. The self-similarity of the process in this case restricts the possible spatial distribution of plasma density to the Gaussian one. However, its initial scales can be chosen in an arbitrary way.

Although the obtained self-similar solution is not general, it includes certain universal laws of plasma expansion, which can be derived directly from the set of equations (1), (2), and (5) without any assumptions about the self-similarity of the solution. One of the laws is the equality of electron and ion average velocities, $\mathbf{u}_e(\mathbf{r}, t)$ and $\mathbf{u}_i(\mathbf{r}, t)$:

$$\mathbf{u}_i(\mathbf{r}, t) = \mathbf{u}_\mathbf{e}(\mathbf{r}, t) \equiv \mathbf{u}(\mathbf{r}, t). \tag{18}$$

Another integral of the systems (1), (2), and (5) is the conservation of full kinetic energy of the ion and electron motions along each direction x_k:

$$W_k \equiv \int \int v_k^2 (m f_e + M f_i)\, d^\nu \mathbf{v}\, d^\nu \mathbf{r} = \text{const}. \tag{19}$$

These integrals [(18) and (19)] allow us to derive differential equations for time dependencies of the plasma density scale lengths (8) from the basic set of equations,

$$d^2 l_k^2 / dt^2 = 2 \tilde{c}_k^2, \qquad \tilde{c}_k^2 \equiv c_k^2 + [\dot{l}_k(0)]^2, \tag{20}$$

where the constants $\tilde{c}_k$ are determined as ratios of kinetic energies, W_k, to the full mass of the plasma.

The solutions of (20) present the universal law of quasineutral plasma expansion (13), which was obtained above for the self-similar case. It is remarkable that this law holds for a plasma bunch with arbitrary initial conditions.

Returning to the obtained exact self-similar solution, it should be mentioned that the ion and electron thermal energies gradually transfer to the energies of their collective motion as the plasma expands. Particles cool in an adiabatic manner; i.e., their thermal velocities reduce in inverse proportion to the corresponding scale length of the plasma bunch (11). It would hardly be surprising that such a result takes place for hot light electrons (when the inequalities (16) are valid) since, in this case, electrons oscillate in the slowly varying potential well. However, the adiabatic cooling is a characteristic peculiarity of the plasma bunch expansion, independent of the value of Zm/M. What is more, the adiabatic low holds for both types of particles, whereas particles of only one type oscillate in the potential φ and the other particles are ejected from the plasma by the electric field.

As follows from Eq. (15), the process of plasma expansion can proceed in two ways, depending on the relationship between the initial values of electron and ion thermal velocities. For the case $V_k^{(e)}(0) > V_k^{(i)}(0)$, the average kinetic energy of electron motion along x_k is reduced and the corresponding energy of ions is increased. When $V_k^{(e)}(0) < V_k^{(i)}(0)$, the average ion kinetic energy is reduced. The characteristic time interval of energy transfer corresponds to the time of ion sound propagation through the region initially occupied by plasma.

Let us emphasize the case of expansion of a plasma bunch with equal initial thermal ion and electron velocities, which corresponds to the absence of any space charge density. The solution obtained in this case is absolutely exact. It shows that the plasma expands with the sonic velocities, which are equal to the initial values of thermal ion and electron velocities. It should be noted that in the latter case an exact solution of Eqs. (1) and (2) can be obtained without restrictions on the electron and ion distribution functions which are imposed by the self-similarity requirement. Accordingly, a time-asymptotic regime of plasma expansion is found to be self-similar, independent of the initial relationship between the spatial and velocity distributions. The self-similar time-asymptotic regime of plasma expansion was also confirmed in numerical simulation [12,13]. This fact suggests that the obtained self-similar solution provides an accurate description of the real process.

In conclusion, an exact self-similar solution to Vlasov equations for electrons and ions has been found analytically within the quasineutral approximation. The obtained results describe the process of plasma expansion for an arbitrary electron distribution over velocities and a large variety of spatial distributions of plasma density in one-, two-, and three-dimensional geometries, including the cases of asymmetric initial spatial distribution of the plasma. The constructed solutions are valid for an arbitrary electron-to-ion mass ratio. Therefore, they can be used to investigate the problem of expansion of an arbitrary two-component plasma (for example, electron-positron or ion-ion plasma).

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