Radiation Transfer Model of Self-Trapping Spatially Incoherent Radiation by Nonlinear Media

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We modify the radiation transfer approach, accounting for refraction, and apply this model to the analysis of propagation of spatially incoherent beams in inertial nonlinear media. For double Gaussian beams, an oscillatory regime of nonlinear diffraction is revealed. An explicit analysis of the oscillations was carried out for logarithmic saturating nonlinearity. We show that "big incoherent solitons," which possess an arbitrary transverse profile of intensity and exist for a wide class of nonlinear media, are steady state solutions of the radiation transfer equation. [S0031-9007(98)07197-X]

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Trapping spatially incoherent laser beams [1] and white light [2] by self-induced waveguides in photorefractive crystals was demonstrated recently. In contrast to self-trapping by conventional Kerr media where giant laser pulses [3,4] are normally required, these observations correspond to milliwatts and nanowatts power of trapped light. A new situation regarding conventional photorefractive spatial solitons [5–7] arose as well. Spatial coherence is necessary to provide replication of the trapped beam intensity pattern by means of a photorefractive response, which is very slow. However, the crystals [1,2] didn't follow fast variations of the incoherent light pattern in either time or space, and only responded to time-averaged local intensity.

The common theoretical approach to the self-trapping occurs in terms of self-similar solutions of the nonlinear wave equation. It is called a spatial soliton [3–7]. For coherent beams such solutions have been found for various nonlinear media, including photorefractive crystals [5–7]. For many of them the areas of stability were analyzed, and different revelations of instability were demonstrated [7]. In particular, an oscillatory regime of propagation for the coherent trapped beam was revealed [7]. A serious complication appears, however, if one would like to use similar terms to discuss trapping spatially incoherent light. The point is that there is no equation that directly applies when looking for a potential soliton-like solution. Indeed, the self-similar profile observed (Refs. [1] and [2]) is the time-averaged intensity. But the wave equation deals with amplitudes, i.e., with instantaneous realizations of speckle patterns, which are certainly not self-similar.

Three different models of incoherent self-trapping were proposed [8–13]. One model operates with a specific function referred to as a coherent density [8,9]. Using this technique, a numerical analysis of fine structure of coherence within the evolving nonlinear waveguide was carried out [9]. The alternative approach treats a light-induced variation of the refractive index as a multimode waveguide [10–12]. For a model of logarithmic nonlinear media, it was shown that a Guassian beam creates a waveguide with transverse modes that after incoherent superposition give the same Gaussian profile of intensity self-consistently [11]. Few-mode waveguides were later studied using the same terms [12]. The authors of the paper [13] proposed an original geometric optics approach utilizing a ray density function [14]. The integral relation between local intensity and ray density was pointed out as a property of invariant propagation. This relation predicts that "big incoherent solitons" can exist for any nonlinear medium and at arbitrary transverse profiles of intensity [13]. These three approaches differ but merge well in their final results. They all are in good agreement with experiments. But even the most general approach [13], which was estimated by its authors as "intuitive, advances predictions" rather than as a quantitative model, treats the self-similar regime only and requires probably deeper background.

In this Letter we address the following problems. First, we point out a local angular spectrum as a relevant characteristic for nonlinear diffraction of an incoherent beam. The local spectrum, coherence density [8,9], and ray density [13] are kindred functions. However, the angular spectrum is an easily observable characteristic, and its spatail evolution can be described in terms of a self-consistent differential equation of the first order that possesses self-similar solutions. We derive this equation as a second step of our analyses. Third, we analyze the evolution for Gaussian beams and show that, in general, they demonstrate oscillatory behavior. Explicit expressions for the dynamics of these oscillations are found for a logarithmic saturating nonlinear medium.

Physics of light trapping in the self-focusing media, $d\varepsilon/dI > 0$, is as follows. The narrow beam creates a waveguide that can hold light inside its boundaries due to the smooth step of the refractive index between illuminated and dark areas. The confinement is effective if a condition $\Delta\varepsilon \geq \Delta\theta^2$ of total internal reflection [14,15] from waveguide flanks is satisfied. Here $\Delta\varepsilon$ is the susceptibility increment in the waveguide center and $\Delta\theta$ is a divergence of the captured beam. The spatially coherent captured beams propagate as the lowest single mode of the induced waveguide, hence light divergence is $\Delta\theta \sim \lambda/a$. This relation specifies the transverse size a of the beam with respect to the wavelength λ and the minimal index modulation $\Delta\varepsilon$ required for self-trapping, $\Delta\varepsilon \geq (\lambda/a)^2$ [16]. If the trapped beam is spatially incoherent, $l_{coh} \ll a$, the corresponding waveguide is a multimode one. The divergence is $\Delta\theta \sim \lambda/l_{coh}$, where l_{coh} is the transverse size of the coherence area, and it is not related to the beam size a. The guiding efficiency in the multimode case is controlled by the ratio $\Delta\varepsilon/\Delta\theta^2$ only. Hence an incoherent self-similar solution could not specify any transverse size or particular profile of intensity of the beam [13]. Such a property is a peculiar one among the family of solitons; normally a type of medium nonlinearity rigorously controls their particular shapes [5–7,16,17].

We are interested in a steady state equation for the spatial evolution of the local angular spectrum $J(\theta, \mathbf{r}, z)$ of the incoherent beam $E(\mathbf{R}, t)\exp(-i\omega t + ikz)$, which propagates along the z axis in a medium with smooth optical inhomogeneities $\varepsilon(\mathbf{R}) = \varepsilon_0 + \delta\varepsilon(\mathbf{R})$. Here the central frequency ω and the wave number $k = \omega n/c$ are introduced; $\varepsilon_0 = n^2$ is the spatially uniform part of the dielectric susceptibility. The equation of interest can be derived in paraxial approximation using the technique found in Refs. [18] and [19].

Any spatially incoherent beam can be treated as a sequence of completely coherent speckle patterns $E(\mathbf{R})$ that are permanently changing in time, $E(\mathbf{R}, t)$. An average time τ_{coh} of the pattern's substitution is the coherence time. We assume that τ_{coh} exceeds the time L/c that it takes for photons to pass through the thickness L of the medium. From then on, we apply the parabolic wave equation for the slow amplitude $E(\mathbf{R}, t)$ that depends on t as a parameter via boundary profile $E(\mathbf{r}, z = 0, t)$:

$$2ik(\partial E/\partial z) + \Delta_\perp E = -(\omega/c)^2 \delta\varepsilon(\mathbf{R}) E(\mathbf{R}, t) . \quad (1)$$

The same equation can be rewritten for the product $B = E^*(\mathbf{r}_2, z, t)E(\mathbf{r}_1, z, t)$ of two amplitudes at different points $\mathbf{r}_1$ and $\mathbf{r}_2$ of the same cross section z as

$$\partial B/\partial z - (i/k)\nabla_{\mathbf{r}}\nabla_{\boldsymbol{\rho}} B = (i/2k)(\omega/c)^2\{\delta\varepsilon(\mathbf{r}_1, z) - \delta\varepsilon(\mathbf{r}_2, z)\}B(\mathbf{r}, \boldsymbol{\rho}, z, t) . \quad (2)$$

If time averaged, the function B gives the spatial correlation function $\langle B(\mathbf{r}, \rho)\rangle$. The middle-point coordinate $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ corresponds to slow spatial variations of the average beam intensity; at $\boldsymbol{\rho} = 0$ it gives just $\langle |E(\mathbf{r})|^2\rangle$. The coordinate $\boldsymbol{\rho} = (\mathbf{r}_1 - \mathbf{r}_2)$ is responsible for fast transverse variations of the complex amplitude for realizations of speckle patterns. The local angular spectrum $J(\boldsymbol{\theta}, \mathbf{r}, z)$ at the spatial location $(\mathbf{r}, z)$ can be defined as a Fourier transform over the fast coordinate ρ for the function B, which is then averaged over time:

$$J = \langle (k/2\pi)^2 \iint d^2\rho B(\mathbf{r}, \boldsymbol{\rho}, z, t)\exp(-ik\boldsymbol{\theta}\boldsymbol{\rho})\rangle .$$

Being integrated over $d^2\theta$, this function gives the local intensity $I(\mathbf{r}, z) = \int d^2\boldsymbol{\theta} J(\mathbf{r}, \boldsymbol{\theta}, z)$.

Variations of optical density are induced by the averaged intensity $I(\mathbf{r}, z)$, and are slow, $|d\delta\varepsilon/dr| \sim \delta\varepsilon/a$, while the scale of decorrelation for $\langle B(\boldsymbol{\rho})\rangle$ is much smaller, $l_{coh} \ll a$. Then the increment of $\delta\varepsilon$ between two points in the right-hand side of Eq. (2) can be replaced by $\boldsymbol{\rho} \cdot \nabla_{\mathbf{r}}\delta\varepsilon$. After this substitution, Fourier transforming, and averaging Eq. (2), one has modified the radiation transfer equation

$$\frac{\partial J}{\partial z} + \left(\boldsymbol{\theta} \cdot \frac{\partial J}{\partial \mathbf{r}}\right) + \frac{1}{2\varepsilon_0}\left(\frac{\partial \delta\varepsilon}{\partial \mathbf{r}} \cdot \frac{\partial J}{\partial \boldsymbol{\theta}}\right) = 0 , \quad (3)$$

which takes into account the refraction at smooth profile $\delta\varepsilon(\mathbf{r}, z)$. The equation obtained seems to be general enough. It is derived for $\delta\varepsilon$ from a general origin and is applicable for both linear and nonlinear media. The only requirements for its derivation are $|\delta\varepsilon| \ll 1$, $|\Delta\theta| \ll 1$, and $l_{coh}/a \ll 1$, which are all very common in optics. If the dispersion of $\delta\varepsilon(\omega)$ is negligible, this equation is applicable to the case of white light. The coefficients there do not depend on wavelength, and one can derive Eq. (3) for each narrow frequency bandwidth where Eq. (1) can be applied and simply sum the results. This summing gives the proper result in the steady state even for nonlinear case since the medium response is time averaging.

We limit our analysis in this Letter to the case of cylindrical symmetry, $J(\mathbf{r}, \boldsymbol{\theta}, z) = J(r, \theta, z)$, where $r = |\mathbf{r}|$, $\boldsymbol{\theta} = |\theta|$, since it is relevant to the experiments [1,2]. We begin with self-similar solutions, and assume that both $\delta\varepsilon(\mathbf{r})$ and $J(\mathbf{r}, \theta)$ do not depend on z. The general solution of the resulting equation can be expressed as a function of the only scalar parameter u, $J(r, \theta) = J(u)$, and

$$u(r, \theta) = [\delta\varepsilon(r)/\varepsilon_0 - \theta^2]/\theta_0^2 . \quad (4)$$

Here, both $J(u)$ and θ_0^2 should be specified by analysis of particular cases.

One can begin with the factorization of the angular and spatial dependencies. A general form of the factorized self-similar solution of Eq. (3) is

$$J(\theta, r) = J_0 \exp[-(\theta/\theta_0)^2 + \delta\varepsilon(r)/(\varepsilon_0\theta_0^2)]$$

since the only option is to use $J(u)$ as an exponent; J_0 and θ_0^2 are constants. This solution can be applied to either a nonlinear or a linear medium that contains the smooth over r and constant over z variation $\delta\varepsilon(r)$; for the nonlinear case it was found earlier (see Ref. [13]). Thus, only the Guassian shape of the angular spectrum for incoherent light provides factorization, i.e., the uniform angular characteristics of radiation over the whole cross section. Otherwise, spatial and angular dependencies cannot be separated.

The factorized solution demonstrates the known effect of light concentration in areas of larger susceptibility

$\delta\varepsilon > 0$. For the linear medium, the parameters J_0 and θ_0 can be arbitrary at any $\delta\varepsilon(r)$ chosen. For instance, in the regions of negative variations of the optical density, $\delta\varepsilon < 0$, the antiguiding effect occurs: The intensity can drop exponentially in the center $r = 0$ if $\Delta\varepsilon/\theta_S^2 < (-1)$. Both guiding and antiguiding behaviors do not specify any profile in the solution $\delta\varepsilon(r)$, but linear self-similar propagation requires nevertheless perfect matching intensity to $\delta\varepsilon(r)$ as $I(r) \propto \exp[\delta\varepsilon(r)/(\varepsilon_0\theta_S^2)]$.

For the nonlinear case, one can integrate the spectrum obtained over the solid angle $d^2\theta = \pi d\theta^2$ to find the local intensity $I(r)$ that creates the waveguide $\delta\varepsilon(r)$. A result given for the factorized solution is a relation between I and $\delta\varepsilon$,

$$\delta\varepsilon = \Delta \ln(I/I_t)\,, \tag{5}$$

where $\Delta = \varepsilon_0\theta_0^2$ and I_t is some constant—a saturation intensity. The relation should hold true irrespective of the particular transverse dependencies $I(r)$ and $\delta\varepsilon(r)$, and specifies, hence, a type of medium nonlinearity, which allows for this solution. Thus, the nonlinearity in Eq. (5) is the only law of medium response which maintains factorized self-similar propagation. The condition of self-similarity in the nonlinear case is that the divergence θ_0 should precisely match the particular amplitude of the nonlinearity, $\theta_0 = \theta_S = (\Delta/\varepsilon_0)^{1/2}$. If this is the case, then any profile of the waveguide satisfies a condition of self-consistent superposition [10–12] for its transverse modes. In particular, dark and bright solitons [5–7] represent the far limits of the factorized solution at $\delta\varepsilon(0) > 0$ and $\delta\varepsilon(0) < 0$, respectively; they can be permanently transformed (one into another) without changing the sign of Δ.

Integration of $J(u)$ over the solid angle gives, at arbitrary nonlinear response, the local intensity $I(r)$ for self-similar solutions in the form

$$I(r) = \pi\theta_0^2 \int_{-\infty}^{\delta\varepsilon(r)/\varepsilon_0\theta_0^2} J(u)du\,. \tag{6}$$

Equation (6) should be identical to the material relation between the variation $\delta\varepsilon$ and the local intensity I that induces this variation. The function $J(u)$ is, thus, specific for any given type of self-focusing nonlinearity. It can be easily found by differentiating the reversed dependence $I(\delta\varepsilon)$ as $J(\delta\varepsilon/\varepsilon_0) = (\varepsilon_0/\pi)dI/d(\delta\varepsilon)$ [13]. Two other examples can be derived from Eq. (6) (and found also in Ref. [13]). For the Kerr media, $\delta\varepsilon = \varepsilon_2 I$, one has a steplike solution, $J(u) = (\varepsilon_0/\pi\varepsilon_2)$ at $u > 0$ and $J(u) = 0$ at $u < 0$. This corresponds to the uniform cone angular spectrum; its half-width varies in space following the local intensity $\theta_{\max}(r) = [\varepsilon_2 I(r)/\varepsilon_0]^{1/2}$, and $I(r)$ can be arbitrary again.

Another example is saturable nonlinearity $\delta\varepsilon(I) = \Delta\varepsilon_m[(I/I_s)/(I + I/I_s)]$. It is relevant to the drift response used in the experiments [1,2] if anisotropy is neglected [5–7]. It could also be interesting for similar experiments with resonant gases or with saturable absorbers. A condition for the time-averaging response then is that the coherence time $\tau_{\rm coh}$ lies in between a population relaxation time I/γ and a dephasing time $I/\delta\omega$ for a saturated optical transition. This self-similar spectrum is not factorized for this solution, and the arbitrary profile $\delta\varepsilon(r)$ is also allowed. But, it should fit $\delta\varepsilon(r)$ in this case as

$$J = I_s/(\pi\theta_0^2)\{(\theta^2/\theta_0^2) + [\Delta\varepsilon_m - \delta\varepsilon(r)]/\Delta\varepsilon_m\}^{-2},$$

where $\theta_0^2 = \Delta\varepsilon_m/\varepsilon_0$. At self-trapping into a bright soliton, the divergence in the center is lower, but grows toward the edges. Thus, all of the profiles for the big incoherent solitons obtained in Ref. [13] turn out to be steady state solutions of the radiation transfer equation.

We will illustrate the potential of Eq. (3) for the simplest case of a "double-Gaussian" beam [19], looking for a solution evolving over z in the form

$$J = J_0 \exp[-\alpha(z)\mathbf{r}^2 - \beta(z)(\mathbf{r}\cdot\theta) - \gamma(z)\theta^2]\,. \tag{7}$$

This corresponds to the local Gaussian distribution $I(\mathbf{r}, z) = [\pi J_0/\gamma]\exp[-\mathbf{r}^2/a^2]$ of intensity at a beam radius $a(z) = (\alpha - \beta^2/4\gamma)^{-1/2}$ that carries total power $W_0 = \pi^2 J_0 a^2/\gamma$. The waveguide profile can be approximated for this case as a parabolic one, neglecting higher order terms over the parameter $[\mathbf{r}/a(z)]^2$. Then, for any type of nonlinearity, the last coefficient in Eq. (3) can be presented as $-p(z)\mathbf{r}$, where $p = (1/a)^2[(I/\varepsilon_0)d\delta\varepsilon/dI]$ is taken in the point $r = 0$. One has three equations for three variables as a result of this approximation:

$$d\alpha/dz = p\beta, \qquad d\beta/dz = 2(p\gamma - \alpha),$$
$$d\gamma/dz = -\beta\,.$$

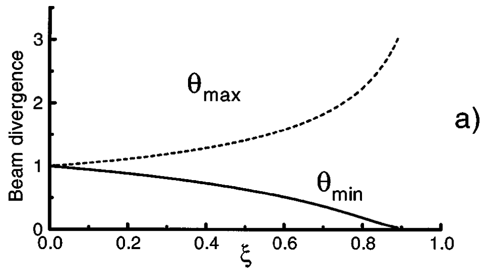
The integral $\gamma\alpha - \beta^2/4 =$ const of this system represents a conservation of the beam power W_0, and, since a z dependence of the parameter $p = p(\alpha, \beta, \gamma)$ is not explicit, this system can be treated as a Hamiltonian one that possesses periodic orbits in its phase space. Hence one has to look for oscillating solutions of this system.

For the nonlinearity in Eq. (5) the waveguide turns out to be a precisely parabolic shape, $p = (\Delta/\varepsilon_0)a^{-2}(z)$, and the system above is precise. It makes sense to look for explicit expressions for this particular case. This system can be reduced then to a single equation,

$$d^2\gamma/dz^2 + 2\alpha_c \ln(\gamma/\gamma_c) = 0\,, \tag{8}$$

where $\alpha_c = (1 - \xi)/a_0^2$ and $\gamma_c = \theta_S^{-2}(1 - \xi)\exp[\xi/(1 - \xi)]$ can be expressed via a minimal radius a_0 of the oscillating beam and the only control parameter ξ of the problem. This value, $0 < \xi < 1$, relates the maximal beam divergence $\theta_{\max}$ to the divergence $\theta_S = (\Delta/\varepsilon_0)^{1/2}$, which provides self-similar propagation for the nonlinearity [Eq. (5)], $\theta_{\max} = \theta_S/(1 - \xi)^{1/2}$.

At $\xi \to 0$, Eq. (8) results in the self-similar solution that corresponds to the beam of an arbitrary radius a_0 and the constant divergence θ_S. At $\xi \ll 1$ both the divergence $\theta(z)$ and the radius $a(z)$ oscillate at a small relative amplitude ξ, but keep the product $a(z)\theta(z)$ constant. Oscillations are around this self-similar solution, and occur at a spatial period $\Delta z_0 = (a_0/\theta_S)\pi\sqrt{2}$. For larger ξ the



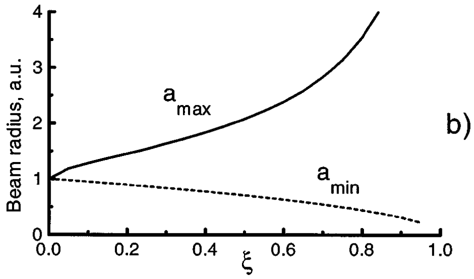


FIG. 1. (a) Minimum and maximum divergence for the oscillating solution of Eq. (8) normalized at the θ_S versus the control parameter ξ. (b) Corresponding dependencies of maximum and minimum beam radius for the same solution.

oscillations are nonlinear, and the period grows as $\Delta z \cong \Delta z_0 \exp[(\xi/2)/(1 - \xi)]$; oscillation swings are illustrated in Fig. 1.

For self-similar solutions of Eq. (3), precise matching beam divergence to the medium response and specific boundary conditions are required. Both are rather hard to implement in experiments. We thus suppose that the oscillating solutions are observed, and the amplitude of the oscillations is controlled by a matching accuracy.

In conclusion, we propose the radiation transfer approach for theoretical analysis of nonlinear self-trapping effects for incoherent radiation. We have shown that big incoherent solitons, which exist for any time-averaging self-focusing nonlinearity and do not specify any particular shape of beam intensity, correspond to steady state solutions of the radiation transfer equation, and we have found spatially oscillating solutions for a double Gaussian beam in logarithmic saturable nonlinearities.

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