Thouless Energy and Correlations of QCD Dirac Eigenvalues

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Eigenvalues and eigenfunctions of the QCD Dirac operator are studied for an instanton liquid partition function. We find that for energy differences δE below an energy scale E_c, identified as the Thouless energy, the eigenvalue correlations are given by random matrix theory. The value of E_c shows a weak volume dependence for eigenvalues near zero and is consistent with a scaling of $E_c \sim 1/L^2$ in the bulk of the spectrum in agreement with estimates from chiral perturbation theory that $E_c/\Delta \approx F_\pi^2 L^2/\pi$ (with average level spacing Δ). For $\delta E > E_c$ the number variance shows a linear dependence. For the wave functions we find a small nonzero multifractality index. [S0031-9007(98)06623-X]

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Random matrix theories have been applied to many aspects of mesoscopic systems (see [1–3] for recent reviews). In particular, eigenvalue correlations have received a great deal of attention in this context. Two important energy scales have been identified: (i) the Thouless energy, defined as the inverse diffusion time of an electron through the sample, i.e.,

$$E_c = \frac{\hbar D}{L^2}, \tag{1}$$

where D is the diffusion constant, and (ii) the energy $\hbar/\tau_e$, where τ_e is the elastic collision time. Eigenvalue correlations on a scale δE can then be classified according to the following three regimes: the ergodic regime for $\delta E < E_c$, the diffusive or Altshuler-Shklovskii [4] regime for $E_c < \delta E < \hbar/\tau_e$, and the ballistic regime for $\delta E > \hbar/\tau_e$ (see [5] for recent work on this topic).

The eigenvalue correlations can be measured conveniently by the number variance, $\Sigma_2(n)$. This statistic is defined as the variance of the number of levels in an interval that contains n levels on average. In the ergodic regime, eigenvalue correlations are given by the invariant random matrix ensembles with $\Sigma_2(n) \sim (2/\beta\pi^2)\log(n)$. In the diffusive regime the situation is more complicated. For a critical system, with a localization length that scales with the size of the sample, it is argued [4] that $\Sigma_2(n) = \chi n$ (with $\chi < 1$), whereas for weaker disorder the expectation is that $\Sigma_2(n) \sim n^{d/2}$. For a critical system, the slope of the number variance has been related to the multifractality index of the wave functions [6].

In this Letter we wish to investigate to what extent such scenarios are realized in QCD. We will investigate eigenvalues of the Euclidean Dirac operator. Because of the $U_A(1)$ symmetry they occur in pairs $\pm\lambda_k$ or are zero. What is of main interest are the eigenvalues near zero which, for broken chiral symmetry, are spaced as $\Delta = \pi/\Sigma V$ (the space time volume is denoted by V). This is based on the Banks-Casher formula [7] according to which the order parameter of the chiral phase transition, Σ, and the spectral density near zero are related by $\Sigma = \pi\rho(0)/V$. It is therefore natural to define the microscopic limit, $V \to \infty$ with $u = \lambda V \Sigma$ kept fixed and the associated microscopic spectral density [8]

$$\rho_S(u) = \lim_{V\to\infty} \frac{1}{V\Sigma}\left\langle \rho\left(\frac{u}{V\Sigma}\right)\right\rangle. \tag{2}$$

There is ample evidence from lattice QCD [9,10] and instanton liquid [11] simulations that $\rho_S(u)$ and other correlators on the scale of individual level spacings [12] are given by chiral random matrix theory (chRMT), i.e., RMT's with the chiral symmetries of the QCD partition function. However, at scales beyond a few eigenvalue spacings in both instanton [11] and lattice QCD [9,10] simulations the Dirac eigenvalues near zero show stronger fluctuations than in chRMT. This indicates the presence of an energy scale in QCD which may be identified as the Thouless energy. The interpretation of spontaneous chiral symmetry breaking as a delocalization transition was made earlier in [13]. By analogy with the Kubo formula, Σ plays the role of the conductivity [13].

Because of the chiral symmetry of the Dirac operator and its spontaneous breaking, the eigenvalue correlations near zero in the ergodic domain are given by the chiral ensembles [8,14]. This is the domain where pion loops can be ignored. Its boundary is thus given by [15,16] a valence quark mass scale m_c where the mass of the associated Goldstone boson, $\sqrt{m_c B}$ (according to the PCAC relation $B = \Sigma/F_\pi^2$, where F_π is the pion decay constant), is of the order of the inverse linear dimension of the box. This relation can be rewritten as [17]

$$m_c = \frac{1}{BL^2}. \tag{3}$$

It is therefore tempting to interpret $1/B$ as the diffusion constant. Because B is large on a hadronic scale ($B \approx 1660$ MeV), the diffusion constant is relatively small. With eigenvalue spacing $\Delta = \pi/\Sigma V$, this condition can be rewritten in dimensionless form as

$$g_c = \frac{m_c}{\Delta} = \frac{F_\pi^2}{\pi} L^2. \tag{4}$$

Here, g_c plays the role of the dimensionless conductivity. Another discussion of F_π in terms of the conductivity is

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given in [18]. In lattice QCD, for an Na^4 lattice, this relation reads $g_c = F_\pi^2 a^2 \sqrt{N}/\pi$. On a 16^4 lattice with a lattice spacing of 0.2 fm this results in a dimensionless Thouless energy of about one level spacing. The above discussion was for sea quark masses much less than the valence quark mass scale m_c. Further subtleties arise in the quenched limit (see [19] for more details).

The chiral random matrix theories for N_f massless quarks in the sector of topological charge ν are defined by the partition function [8,14]

$$Z_{N_f,\nu}^{\beta} = \int DW \det{}^{N_f}\begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} e^{-n\beta \operatorname{Tr} V(W^\dagger W)}, \quad (5)$$

where W is a $n \times (n + \nu)$ matrix. The parameter $2n$ is identified as the dimensionless volume of space time. In this paper we only consider the chiral Gaussian unitary ensemble (chGUE) with complex matrix elements W ($\beta = 2$) and a Gaussian potential $V(x) = \Sigma^2 x$. In this case $\rho_S(u)$ and all other correlation functions have been derived analytically [10,14,20]. We note only that in the bulk of the spectrum the correlations are given by the invariant RMT's. It can be shown [21,22] that, for $\beta = 2$, $\rho_S(u)$ and other correlators do not depend on the potential $V(x)$. The application of chRMT to QCD has been put on a firm foundation by these and other universality proofs [23–25]. Whether or not QCD is in this universality class is a dynamical question that can only be proven by explicit numerical simulations.

Our calculations will be performed using the instanton liquid model. In this model the gauge field configurations are given by a superposition of instantons. The Euclidean QCD partition function is then approximated by

$$Z_{\text{inst}} = \int D\Omega \, \det{}^{N_f}(D + m) e^{-S_{\text{YM}}}, \quad (6)$$

where the integral is over the collective coordinates of the instantons and the Dirac operator is denoted by D. For each instanton we have 12 collective coordinates (for three colors). The Yang-Mills action is denoted by S_{YM}. The fermion determinant is evaluated in the space of the fermionic zero modes of the instantons. In our calculations we use the standard instanton density $N/V = 1$ (in units of fm^{-1}). For further discussion of this partition function, which obeys the flavor and chiral symmetries of the QCD partition function, we refer to [26].

The partition function (6) is evaluated by means of a Metropolis algorithm. We perform on the order of 10 000 sweeps for each set of parameters. The eigenvalues and eigenvectors of the Dirac operator are calculated by means of standard diagonalization procedures. In this Letter we restrict ourselves to $N_f = 0$. This is a natural choice because our results are compared with ideas from disordered mesoscopic systems where no fermion determinant is present, and, moreover, it allows us to study much larger volumes.

In order to separate the fluctuations of the eigenvalues from the average spectral density, the spectrum is unfolded such that the unfolded spectrum has unit average level spacing. All our spectral observables are calculated with the unfolded spectrum.

In Figs. 1 and 2 we show the number variance $\Sigma_2(n)$ versus n for various total numbers of instantons with instanton density $N/V = 1$. The chGUE result for $\Sigma_2(n)$ [10] is depicted by the solid curve. In both figures, the upper figure (with only two volumes) is a blown-up version of the lower figure. In Fig. 1, $\Sigma_2(n)$ is calculated for the interval starting at $\lambda = 0$. Figure 2 represents the number variance in the bulk of the spectrum obtained from an interval that is symmetric about the average positive unfolded eigenvalue. In both cases we observe a clear transition point n_c below which the number variance is given by RMT. In Fig. 1, the value of the crossover point, $n_c \approx 2$, depends only weakly on the total number of instantons (or the volume). This is not in agreement with the theoretical expectation (4) that $n_c \approx F_\pi^2 \sqrt{V}/\pi$ in four dimensions. For $N/V = 1\ \text{fm}^{-4}$ the value of n_c for N instantons is given by $n_c \approx 0.07\sqrt{N}$ which is on the order of the results found in Fig. 1. However, in the bulk of the spectrum (Fig. 2), the value of n_c is consistent with a $\sqrt{V}$ scaling but the numerical constant appears to be larger than the above estimate. This result is in agreement with

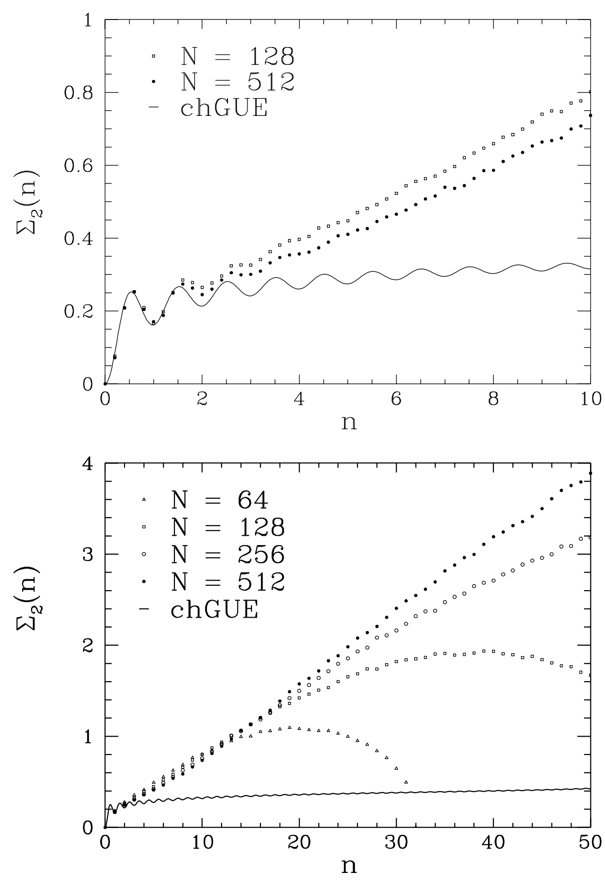

FIG. 1. The number variance $\Sigma_2(n)$ versus n approximation for an interval starting at $\lambda = 0$. The total number of instantons is denoted by N.

the finding that correlations of lattice QCD eigenvalues are given by RMT up to distances of more than 100 spacings [12,27].

Beyond the crossover point the number variance shows a linear behavior with a slope $\chi \approx 0.08$ for eigenvalues near zero and $\chi \approx 0.04$ in the bulk of the spectrum. The downward trend of the curves for larger values of n is a well understood finite size effect. This prevents us from saying more about the ballistic regime, an energy scale of roughly the inverse distance between instantons.

In the ergodic regime we expect that eigenvalue correlations are given by the chiral random matrix ensembles. Indeed, both the microscopic spectral density up to two level spacings and the nearest neighbor spacing distribution are in perfect agreement with the chGUE.

The number of significant components of the wave function is measured by the participation ratio. Its inverse is defined as

$$I_2(\lambda) = \left\langle \sum_k |\psi_k(\lambda)|^4 \right\rangle . \tag{7}$$

The $\psi_k(\lambda)$ are the normalized N-component eigenfunctions (corresponding to eigenvalue λ) of the Dirac operator in the space of the fermionic zero modes of the individual instantons. Results for $NI_2(\lambda)$ as a function of λ are depicted in Fig. 3 (upper). The general impression is that the wave functions are extended with an inverse participation ratio that is not too different from the random matrix result (full line). The eigenfunctions corresponding to small and large eigenvalues appear to be somewhat more localized. A more definitive result for the character of the wave functions follows from the scaling behavior of $I_2(\lambda)$ with the volume. A double logarithmic plot of $NI_2(\lambda)$ versus N is shown in Fig. 3 (lower). Results are given for both the energy intervals $[0.1, 0.2]$ (open circles) and $[0.7, 0.8]$ (full circles). The first region corresponds to a part of the spectrum where the number variance shows a linear behavior for the volumes shown in Fig. 1, and the second region corresponds to the bulk of the spectrum. The multifractality index η is defined by [6]

$$I_2 \sim V^{\eta/d - 1}. \tag{8}$$

According to [6] the value of $\eta = 2\chi d$ (where χ is the slope of the linear piece in the number variance) in the critical domain. From the volume dependence of I_2 shown in Fig. 3 (lower) we find values for η/d of about 0.02 and 0.04 for the intervals $[0.1, 0.2]$ and $[0.7, 0.8]$, respectively. These values are well below the theoretical result of 2χ. Apparently, our ensemble of instantons is

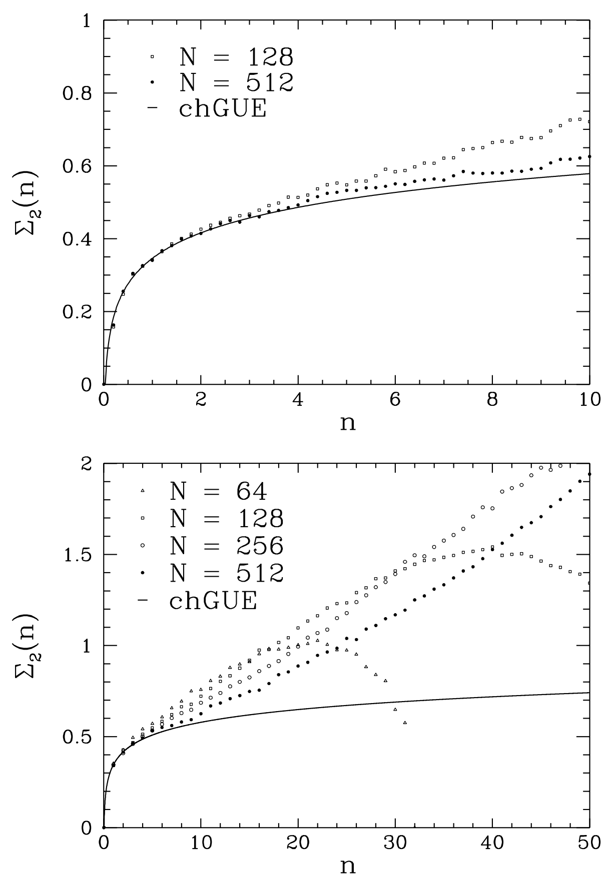

FIG. 2. The number variance $\Sigma_2(n)$ versus n in the bulk of the spectrum.

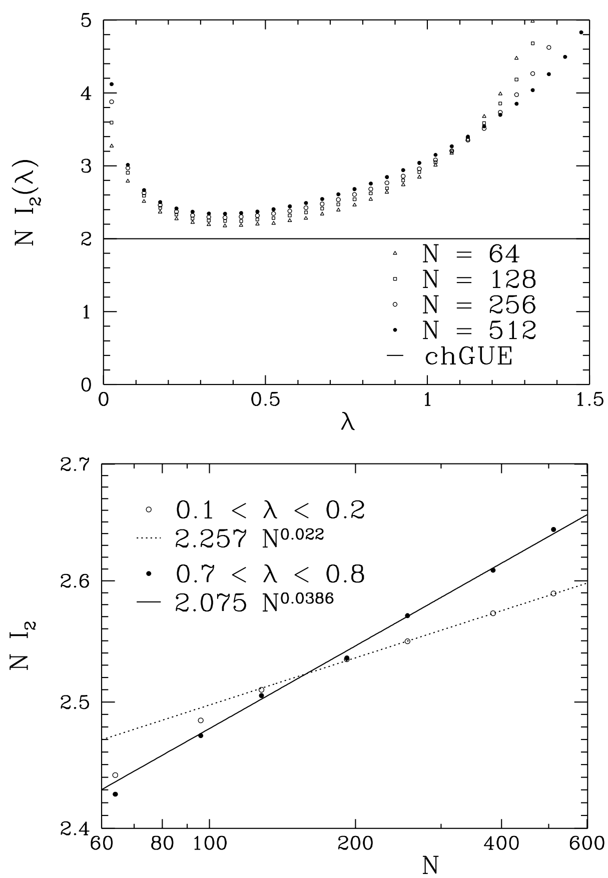

FIG. 3. The inverse participation ratio times N versus the corresponding Dirac eigenvalues (upper), and the scaling behavior versus N (lower) for two energy intervals.

not in the critical region. Our results should be contrasted with Wilson lattice QCD Dirac eigenfunctions which were found to be localized [28]. We have no explanation for this discrepancy.

In conclusion, we have identified an energy scale below which the eigenvalue correlations of the QCD Dirac operator are given by chRMT. In analogy with the theory of mesoscopic systems, this scale will be called the Thouless energy. For eigenvalues near zero we find a Thouless energy that only shows a weak volume dependence, whereas for eigenvalues in the bulk the Thouless energy scales roughly with the square root of the volume in agreement with theoretical prediction.

For energy scales beyond the Thouless energy a linear n dependence of the number variance is found. Our wave functions show a small nonzero multifractality index which does not obey the relation derived for critical mesoscopic systems. Interesting connections with the scalar susceptibility and quenched chiral perturbation theory will be discussed elsewhere [29].

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Note added.—After the completion of this work we received a paper by R. Janik *et al.* [30] in which similar ideas were discussed. In that work the diffusion constant is related to diffusion in a $4 + 1$ dimensional space time.

[1] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[2] C. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
[3] G. Montambaux, cond-mat/9602071.
[4] B. Altshuler, I. Zharekeshev, S. Kotochigova, and B. Shklovskii, Zh. Eksp. Teor. Fiz. **94**, 343 (1988) [Sov. Phys. JETP **67**, 625 (1988)].
[5] A. Altland and Y. Gefen, Phys. Rev. Lett. **71**, 3339 (1993); D. Braun and G. Montambaux, Phys. Rev. **52**, 13 903 (1995); A. Aronov and A. Mirlin, Phys. Rev. B **51**, 6131 (1995); Y. Fyodorov and A. Mirlin, Phys. Rev. B **51**, 13 403 (1995); T. Guhr and A. Mueller-Groeling, cond-mat/9702113; V. Kravtsov, I. Lerner, B. Altshuler, and A. Aronov, Phys. Rev. Lett. **72**, 888 (1994); A. Altland, Y. Gefen, and G. Montambaux, Phys. Rev. Lett. **76**, 1130 (1996); N. Argaman, Y. Imry, and U. Smilansky, Phys. Rev. B **47**, 4440 (1993); K. Frahm, T. Guhr, and A. Mueller-Groeling, cond-mat/9801298.
[6] J. Chalker, V. Kravtsov, and I. Lerner, JETP Lett. **64**, 386 (1996); V. Kravtsov and I. Yurkevich, cond-mat/9612036.
[7] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[8] E. Shuryak and J. Verbaarschot, Nucl. Phys. **A560**, 306 (1993).
[9] M. Berbenni-Bitsch, S. Meyer, A. Schäfer, J. Verbaarschot, and T. Wettig, Phys. Rev. Lett. **80**, 1146 (1998); R. Pullirsch, K. Rabitsch, T. Wettig, and H. Markum, hep-ph/9803285.
[10] J. Z. Ma, T. Guhr, and T. Wettig, hep-lat/9712026.
[11] J. Verbaarschot, Nucl. Phys. **B427**, 434 (1994).
[12] M. Halasz and J. Verbaarschot, Phys. Rev. Lett. **74**, 3920 (1995).
[13] E. Shuryak, Nucl. Phys. **B302**, 599 (1988); D. Diakonov, hep-ph/9602375; A. V. Smilga, Phys. Rep. **291**, 1 (1997).
[14] J. Verbaarschot, Phys. Rev. Lett. **72**, 2531 (1994); Phys. Lett. B **329**, 351 (1994); Nucl. Phys. **B426[FS]**, 559 (1994).
[15] J. Gasser and H. Leutwyler, Phys. Lett. B **188**, 447 (1987).
[16] H. Leutwyler and A. Smilga, Phys. Rev. D **46**, 5607 (1992).
[17] J. Verbaarschot, Phys. Lett. B **368**, 137 (1996).
[18] J. Stern, hep-ph/9801282.
[19] J. Osborn and J. Verbaarschot, hep-ph/9803419.
[20] J. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993).
[21] G. Akemann, P. Damgaard, U. Magnea, and S. Nishigaki, Nucl. Phys. **B487[FS]**, 721 (1997).
[22] E. Brézin, S. Hikami, and A. Zee, Nucl. Phys. **B464**, 411 (1996).
[23] P. Damgaard, hep-th/9711110; hep-th/9711047; G. Akemann and P. Damgaard, hep-th/9801133; hep-th/9802174; S. Nishigaki, P. Damgaard, and T. Wettig, hep-th/9803007.
[24] T. Guhr and T. Wettig, Nucl. Phys. **B506**, 589 (1997).
[25] A. Jackson, M. Sener, and J. Verbaarschot, Nucl. Phys. **B479**, 707 (1996).
[26] T. Schäfer and E. Shuryak, Rev. Mod. Phys. **70**, 323 (1998).
[27] T. Guhr (private communication).
[28] K. Jansen and C. Liu, Nucl. Phys. Proc. Suppl. **53**, 974 (1997).
[29] J. Osborn, D. Toublan, and J. Verbaarschot, hep-th/9806110 (to be published).
[30] R. Janik *et al.*, preceding Letter, Phys. Rev. Lett. **81**, 264 (1998).