Disordered *XY* Models and Coulomb Gases: Renormalization via Traveling Waves

David Carpentier and Pierre Le Doussal

CNRS-Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 Rue Lhomond, 75231 Paris, France*

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We present a novel renormalization approach to 2D random *XY* models using direct and replicated Coulomb gas (CG) methods. By including fusion of environments (charge fusion in the replicated CG), the distribution of local disorder is found to obey a Kolmogorov nonlinear equation (KPP) with traveling wave solutions. At low T and weak disorder it yields a glassy *XY* phase with broad distributions and precise connections to random energy models. Finding marginal operators at the disorder-induced transition is related to the front velocity selection problem in KPP equations that yield new critical behavior. The method is applied to critical random Dirac problems. [S0031-9007(98)07150-6]

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Two dimensional random systems have attracted considerable recent interest in domains ranging from localization in quantum Hall systems to vortices in superconductors. In the context of localization, progress was made to characterize the multifractal statistics of 2D wave functions using random Dirac models [1,2], extending previous studies in 1D [3]. On the other hand, the glassy properties of vortex phases with disorder was investigated using random *XY* models. While these lines of studies have developed in an apparently disconnected way, they lead to similar proposals [1,4] that remarkable connections exist between the large fluctuation properties of these systems and solvable disordered models [5,6]. To study these connections further, consistent renormalization (RG) techniques are needed. Our aim is to develop such an approach, which, as in other glassy systems, e.g., in 1D [7], requires a proper treatment of broad distributions. Here we focus on the random gauge *XY* model, whose phase diagram, studied long ago [8], has recently been corrected [4,9–13]. We discuss at the end related random Dirac problems.

Topological defects (vortices) of 2D *XY* models can be described as integer charges with Coulomb interactions. To study the resulting Coulomb gas (CG) in the presence of disorder, conventional RG methods [8,14] use a perturbative expansion in the charge fugacity ($y = e^{-\beta\mu}$ with μ the chemical potential) which is assumed to be *uniform over the system*. However, as we show here, in a disordered environment *charge fugacities* strongly fluctuate from site to site at low temperature, invalidating this assumption. The aim of this Letter is thus to develop a novel approach which allows one to treat site dependent charge fugacities, denoted $z_{\mathbf{r}}$, by following their probability distribution under renormalization. By studying this distribution we uncover a transition of a new nature in the random gauge *XY* model at low temperature. Beyond a critical disorder strength, the low temperature quasi-ordered *XY* phase (which extends down to $T = 0$) becomes unstable to the proliferation of defects (vortices). This topological transition is peculiar, as these defects are disorder induced and frozen in rare favorable regions where $z_{\mathbf{r}}$ is of order 1 ($\mu \sim 0$), while most regions are unfavorable ($z \sim 0$). The density $P(z \sim 1)$ of these favorable regions will emerge naturally as the appropriate perturbative parameter, which decreases with the scale below the transition and increases above. It corresponds to the tail of the distribution of fugacities which becomes broad at low T.

Remarkably, the RG equation we find for the distribution of fugacities turns out to be a nonlinear equation which appears in many contexts, the Kolmogorov (KPP) equation [15]. It is known to admit traveling waves solutions, whose velocity selection problem [15] is under current interest [16]. The velocity of the front solutions determines the increase or decrease of $P(1)$, i.e., the phase diagram. Interestingly, universality in the leading corrections to the velocity [15,16] nicely translates into the RG universality around the disorder driven transition. Furthermore, via this KPP equation, a precise connection is found between the charge fugacity distribution and the free energy distribution of a solvable disordered model: the directed polymer (DP) on a Cayley tree [6]. Finally, restriction to the single charge sector yields a RG derivation of the multifractal properties of the critical Dirac wave function.

As in the pure case, the stability of the weak disorder *XY* phase can be correctly inferred by approximate RG studies based on *dipole energies* [4,12,13] instead of the random *charge fugacities* introduced here. Thus our phase diagram has the same topology as the one of [4,12,13], even though precise study of the low temperature regime and transition requires the new method defined here.

The 2D square lattice *XY* model with random phases [8] is defined by its partition sum $Z[A] = \prod_i \int_{-\pi}^{\pi} d\theta_i e^{-\beta H[\theta,A]}$ with

$$\beta H[\theta, A] = \sum_{\langle i,j \rangle} V(\theta_i - \theta_j - A_{ij}) \qquad (1)$$

and $V(\theta) = -\frac{K}{\pi}\cos(\theta)$, $K = \beta J$, $\beta = 1/T$. The A_{ij} are independent Gaussian random gauge fields, with $\overline{A_{ij}^2} = \pi\sigma$. The Villain form [17,18] of this model, which we study in this paper, can be transformed into a CG with

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integer charges defined on the sites $\mathbf{r}$ of the dual lattice with $Z[V] = \sum_{\{n_\mathbf{r}\}} e^{-\beta H}$ and

$$\beta H = -K \sum_{\mathbf{r}\neq\mathbf{r}'} n_\mathbf{r} G_{\mathbf{r}\mathbf{r}'} n_{\mathbf{r}'} + \sum_\mathbf{r} n_\mathbf{r} V_\mathbf{r} , \qquad (2)$$

where $G_\mathbf{k}^{-1} = \frac{1}{\pi}[2 - \cos(k_x a) - \cos(k_y a)]$ is the lattice Laplacian. The bare disorder potential, $V_\mathbf{r} = \frac{K}{\pi} G_{rr'} (\nabla \times A)_{\mathbf{r}'}$ is Gaussian with logarithmic long range correlations $\overline{V_\mathbf{k} V_{-\mathbf{k}}} = 2\sigma K^2 G_\mathbf{k}$. The usual continuum approximation with (integer) charges of hard core a and fugacities $y = e^{-\gamma K}$ of this lattice model is obtained by using the asymptotic form $G_{rr'} \approx (\ln|r - r'|/a + \gamma)(1 - \delta_{rr'})$. Here, the perturbative expansion of $Z[V]$ in y, valid in the dilute limit uniformly over the system, fails.

Let us first sketch the direct RG method suited to the present case where disorder favors some regions, resulting in a site dependent local fugacity $z_\mathbf{r}$. Our expansion captures the limit where the fugacity is negligible almost everywhere except in a few *rare favorable regions*. This is achieved by following the local disorder distribution which is *not Gaussian*, a novel feature from all previous approaches. We find that the disorder $V_\mathbf{r} = V_\mathbf{r}^> + v_\mathbf{r}$ naturally splits into two parts, a *long range correlated Gaussian* part $V_\mathbf{r}^>$ with logarithmic correlator $\overline{(V_\mathbf{r}^> - V_{\mathbf{r}'}^>)^2}^a = 4\sigma K^2 \ln(|\mathbf{r} - \mathbf{r}'|/a)$ and a *local non-Gaussian* part $v_\mathbf{r}$ which defines the *local fugacity variables* $z_\pm^\mathbf{r} = y\exp(\pm v_\mathbf{r})$ for ± 1 charges [19] which have only short range correlations. The RG equation for the distribution $P(z_+, z_-)$ of local environments is obtained from the following two contributions: (i) "*Rescaling*": upon coarse graining $a \to \tilde a = a e^{dl}$, $V^>$ produces a Gaussian additive contribution to v; from $\overline{(V_\mathbf{r}^> - V_{\mathbf{r}'}^>)^2}^a \equiv \overline{(V_\mathbf{r}^> - V_{\mathbf{r}'}^>)^2}^{\tilde a} + \overline{(dv_\mathbf{r} - dv_{\mathbf{r}'})^2}$ one gets $z_\pm^\mathbf{r} \to z_\pm^\mathbf{r} e^{Kdl \pm dv_\mathbf{r}}$ with $\overline{dv_\mathbf{r} dv_{\mathbf{r}'}} = 2\sigma K^2 dl\, \delta_{\mathbf{r},\mathbf{r}'}$. (ii) "*Fusion of environments*": upon the change of cutoff, as illustrated in Fig. 1, two regions with fugacities $z_\pm^{\mathbf{r}'}, z_\pm^{\mathbf{r}''}$ are replaced by a single region at $\tilde{\mathbf{r}} = \frac{1}{2}(\mathbf{r}' + \mathbf{r}'')$ of effective fugacities $\tilde z_\pm = (z_\pm^{\mathbf{r}'} + z_\pm^{\mathbf{r}''})/(1 + z_+^{\mathbf{r}'} z_-^{\mathbf{r}''} + z_-^{\mathbf{r}'} z_+^{\mathbf{r}''})$ obtained from the relative weight W_+/W_0 of a charge 1 configuration (either in $\mathbf{r}'$ or $\mathbf{r}''$) versus a neutral one (either no charge or a dipole). (i) and (ii) yield

$$\partial_l P(z_+, z_-) = \mathcal{O} P - 2P(z_+, z_-) + 2\left\langle \delta\left(z_+ - \frac{z_+' + z_+''}{1 + z_-' z_+'' + z_+' z_-''}\right) \delta\left(z_- - \frac{z_-' + z_-''}{1 + z_-' z_+'' + z_+' z_-''}\right)\right\rangle_{P'P''}, \qquad (3)$$

where $\langle A \rangle_{P'P''}$ denotes $\int_{z_\pm', z_\pm''} A P(z_+', z_-') P(z_+'', z_-'')$ and $\mathcal{O} = K(2 + z_+\partial_{z_+} + z_-\partial_{z_-}) + \sigma K^2 (z_+\partial_{z_+} - z_-\partial_{z_-})^2$ is the diffusion operator [20].

A systematic way to study this problem is to introduce replicas. Starting from (2) we represent Z^m as the partition sum of a CG with *m-vector* charges $n_\mathbf{r}^b$ living on the dual lattice sites. Averaging over disorder, and taking the continuum limit we obtain the m-vector (hard core) CG of partition sum expanded in power of the *vector fugacity* $Y_\mathbf{n}$:

$$\overline{Z^m} = 1 + \sum_{p\geq 2} \sum_{\mathbf{n}_1 \ldots \mathbf{n}_p} \int_{\mathbf{r}_1 \ldots \mathbf{r}_p} Y_{\mathbf{n}_1} \ldots Y_{\mathbf{n}_p} \prod_{i \neq j} \left| \frac{r_i - r_j}{a} \right|^{n_i^b K_{bc} n_j^c},$$

with $K_{bc} = K\delta_{bc} - \sigma K^2$, all integrals being restricted to $|r_i - r_j| > a$, and the sum is over all distinct neutral configurations $\sum_\mathbf{r} n_\mathbf{r}^b = 0$. $Y_\mathbf{n}$ is a function of the replicated charge $\mathbf{n} = (n^1, \ldots, n^m)$ with bare value $Y_\mathbf{n} \approx e^{-\gamma n_b K^{bc} n_c}$. Since $K_{b\neq c} \neq 0$, one cannot restrict to single nonzero component charges [13], as it leads to the erroneous results of [8] at low temperature. However, we stress that this quadratic form for $Y_\mathbf{n}$, which results from the Gaussian nature of the bare disorder, is *not* preserved by the RG as shown below. We now perform the RG analysis of the m-vector CG, extending the scalar case [21], leaving the preceding form unchanged with [18,22]

$$\partial_l K_{bc}^{-1} = d' \sum_{n\neq 0} n^b n^c Y_\mathbf{n} Y_{-\mathbf{n}} , \qquad (4a)$$

$$\partial_l Y_{\mathbf{n}\neq\mathbf{0}} = (2 - n^b K_{bc} n^c) Y_\mathbf{n} + d \sum_{\mathbf{n}'\neq 0,\mathbf{n}} Y_{\mathbf{n}-\mathbf{n}'} Y_{\mathbf{n}'} . \qquad (4b)$$

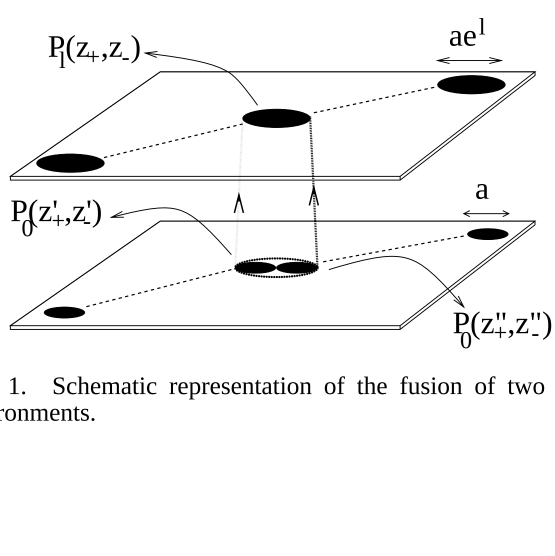

FIG. 1. Schematic representation of the fusion of two local environments.

Rescaling and annihilation of opposite replica charges separated by $a \leq |\mathbf{r}_i - \mathbf{r}_j| \leq a e^{dl}$ gives the first term of (4b) and (4a). The second term of (4b), which comes from *fusion of two replica charges* as usual in *vector* CG, was absent in [4,12,13] but is crucial for the consistency of the RG to order $Y_\mathbf{n}^2$. We take the point of view, as discussed below, that the set of $Y_\mathbf{n}$ encodes the full scale dependent distribution $P(z_+, z_-)$ of local disorder. Remarkably, the correspondence between $P(z_+, z_-)$ and $Y_\mathbf{n}$ emerges when performing the analytical continuation $m \to 0$ of (4) which we now present. To capture the most relevant operators it is sufficient to consider $Y_\mathbf{n}$ with $n^b = 0, \pm 1$ in each replica [19], which, using replica permutation symmetry, depends only on the numbers $n_\pm$ of ± 1 components of $\mathbf{n}$. This leads to the general parametrization in terms of a function

$\Phi(z_+, z_-)$:

$$Y_{\mathbf{n}} = \langle z_+^{n_+} z_-^{n_-} \rangle_\Phi = \left\langle \prod_b [\delta_{n^b,0} + z_+ \delta_{n^b,+1} + z_- \delta_{n^b,-1}] \right\rangle_\Phi ,$$

where $\langle \ldots \rangle_\Phi = \int_{z_\pm} \ldots \Phi(z_+, z_-)$. After some combinatorics [23,24] the limit $m \to 0$ of (4b) can be rewritten *equivalently* as Eq. (3) for $P = \Phi/(\int_{z_+,z_->0} \Phi)$, which is then naturally interpreted as a probability distribution. The first term in (4b) gives the diffusion contribution $\mathcal{O}P$, and the second term in (4b) yields the term of fusion of environments [20] in (3). Finally, with the same definitions, (4a) yields the renormalization for K and σ from screening, which together with (3) form our complete set of RG equations [22]

$$\frac{dK^{-1}}{dl} = \frac{4d'}{d^2} \left\langle \frac{z'_+ z''_- + z'_- z''_+ + 4 z'_+ z''_- z'_- z''_+}{(1 + z'_+ z''_- + z'_- z''_+)^2} \right\rangle_{P'P''}, \quad (5a)$$

$$\frac{d\sigma}{dl} = \frac{4d'}{d^2} \left\langle \frac{(z'_+ z''_- - z'_- z''_+)^2}{(1 + z'_+ z''_- + z'_- z''_+)^2} \right\rangle_{P'P''}. \quad (5b)$$

We have also studied this problem and obtained (3) and (5) using a second method *without replicas*. It also allows one to justify physically the expansion presented previously in the fugacity $Y_{\mathbf{n}}$ of replica charges. Technically, it consists in introducing [24] an *expansion of physical quantities in the number of points*, which for the free energy $F[V] = -T \ln Z[V]$ as a functional of the disorder takes the form

$$F[V] = \sum_{\mathbf{r}_1 \neq \mathbf{r}_2} f^{(2)}_{\mathbf{r}_1,\mathbf{r}_2}[V] + \sum_{\mathbf{r}_1 \neq \mathbf{r}_2 \neq \mathbf{r}_3} f^{(3)}_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3}[V] + \ldots, \quad (6)$$

where by definition $f^{(k)}_{\mathbf{r}_1,\ldots,\mathbf{r}_k}$ depends only on $V(\mathbf{r})$ at points $\mathbf{r}_i$, $i = 1, \ldots, k$. The first term in (6) corresponds to the independent dipole approximation [12], and the second term takes into account contributions from triplets of sites. Equation (6) is identical term by term to the expansion in the vector fugacity $Y_{\mathbf{n}}$ in the replica method used above. It corresponds to a new expansion in the small density of favorable regions $P_l(z \sim 1)$, equivalent to a small density of vector charges on the replicated CG. We have performed (one loop) renormalization of (6): upon coarse graining, each $f^{(k)}$ corrects $f^{(k-1)}$, which is taken into account consistently for all k by the fusion term in (3).

Since we are performing an expansion in the density of rare favorable regions $P_l(z \sim 1)$, it is consistent to discard the $z'_+ z''_-$ terms in the denominators in (3) [25,26]. This leads to a closed equation for a single fugacity distribution $P(z) = \int_{z_-} P(z, z_-)$ which, using the parametrization $G_l(x) = 1 - \langle \exp(-z e^{-\beta(x - E_l)}) \rangle_{P_l(z)}$ where $E_l = \int_0^l J_l dl$, $\beta = 1/T$, can be rewritten as

$$\tfrac{1}{2} \partial_l G = D_l \partial_x^2 G + (1 - G) G . \quad (7)$$

The diffusion coefficient is $D_l = \frac{1}{2} \sigma_l J_l^2$ and by construction $G_l(-\infty) = 1$ and $G_l(+\infty) = 0$. Remarkably, for constant D this is the much studied KPP equation, which describes diffusive invasion of an unstable state ($G = 0$) by a stable one ($G = 1$). It also appears in the solution of the problem of the directed polymer on a disordered Cayley tree [6], where G parametrizes the free energy distribution. It is known [15,16] that $G_l(x)$ converges at large l towards traveling waves solutions $h(x - m_l)$ selected by the behavior at infinity of $G_{l=0}(x) \sim e^{-\beta x}$. This implies $P_l(z) \to z^{-1} p(\ln z - \beta X_l)$ with $X_l = m_l - E_l$ ($X_l < 0$ in the XY phase; see Fig. 2).

We first find at low T, $\sigma < \sigma_c$ an XY phase as in Fig. 2 (K, σ converge to K_R, σ_R). The typical z goes to zero but P develops a broad tail up to $z \sim O(1)$. While in this phase and at criticality the concentration of rare favorable regions $P_l(1)$ decreases, it eventually increases at large l in the disordered phase $\sigma_c \lesssim \sigma$. In the XY phase, one must distinguish two different tails in $P_l(z)$. As shown in Fig. 2, the bulk of the distribution (typical values) is located around $z_{\text{typ}} \sim e^{\beta X_l}$. It corresponds to the *front region* which has a tail of size $\sqrt{l}$ ahead of the front. From the velocity selection studies [15,16] for $T > T_g = J\sqrt{\sigma/2}$ we find the front position $m_l \sim 2(\beta^{-1} + D\beta) l$. For $T < T_g$ the velocity freezes with $m_l = \sqrt{D}\,[4l - \frac{3}{2} \ln l + O(1)]$. This corresponds to $P_l(z) \sim z^{-(1+\mu)}$ within the tail of the front, with $\mu = T/T_g < 1$. Thus for $T < T_g$ the distribution function of $\ln z$ travels at the *relative velocity* $v = \partial_l X_l = J(\sqrt{8\sigma} - 1)$, which determines the phase diagram (see Fig. 3): it is negative [decrease of $P_l(z \sim 1)$] in the low T XY regime, positive for $\sigma \geq \sigma_c$. Furthermore, at low T, there is also a *far tail* ahead $\sim l$ of the front which corresponds to rare events $z \sim 1$, of small probability $P_l(1)$, but which dominate average correlations (and thus $\partial_l K$ and $\partial_l \sigma$). The linearized KPP equation, valid in this region, leads to $P_l(z) \sim P_l(1) z^{-(1+\overline{\mu})}$ with $\overline{\mu} = T/T^* < 1$ and to $P_l(1) \sim e^{(2 - 1/4\sigma) l}$, for $T < T^* = 2\sigma J$.

In the high T regime of the XY phase, $P_l(z)$ is not so broad, and one recovers from (7) the usual RG result [8] $\partial_l y = (2 - K + \sigma K^2) y$ for the average fugacity $y_l = \langle z \rangle_l < +\infty$ ($\sim z_{\text{typ}}$ for $T > T_g$), using $G_l(x) \sim e^{-\beta(x - E_l)} \langle z \rangle_{P_l}$ at large x.

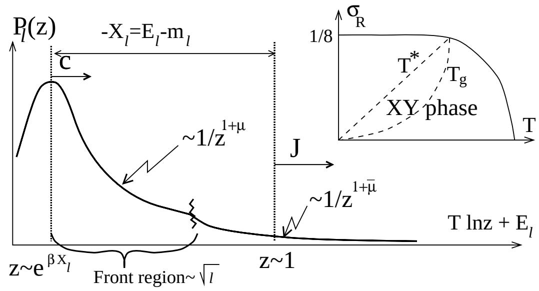

FIG. 2. Scale dependent distribution $P_l(z)$ and its two tail regions for $T < T_g$. Inset: phase diagram.

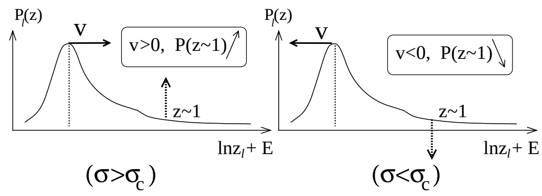

FIG. 3. Relation between the sign of the relative front velocity $v = \partial_l X_l$ and the increase or decrease of $P_l(z \sim 1)$.

The critical behavior at the transition from the XY to the disordered phase is determined by *the front region*, since the velocity $\partial_l X_l$ vanishes. At $T = 0$, we obtain from the *universal* corrections [15,16] to the velocity: $\partial_l X_l = J(\sqrt{8\sigma} - 1) - 3\sqrt{D}/2l + O(l^{-3/2})$ a projection of the RG flow on the plane $\sigma \sim \sigma_c = \frac{1}{8}$ and $g \sim P_l(1)$ [27]

$$\partial_l g = \left(16(\sigma - \sigma_c) - \frac{3}{2l}\right)g; \qquad \partial_l(\sigma - \sigma_c) = g^2,$$

yielding $g_l \sim l^{-3/2}$ at criticality and a correlation length $\xi \sim e^{1/|\sigma - \sigma_c|}$. This new universality class [24] is different from KT and from the prediction of [4,13]. Note that although most details of $P(z_\pm)$, e.g., its bulk, depend on the cutoff procedure (and fusion rule ...), here the universality appears in a remarkable way. It comes from the independence of the velocity and the front tail (which also determine the relevant operators) on the precise form $F[G]$ of the nonlinear term in (7) (see [15]).

Finally, the $E = 0$ critical wave function of 2D Dirac fermions in a random magnetic field B [1–3] satisfies $|\psi_0(\mathbf{r})|^2 = e^{-V_\mathbf{r}}/Z$ with $B = -\frac{1}{2}\nabla^2 V_\mathbf{r}$. It is thus related to the partition function of a *single charge* $Z = \sum_r e^{-V_r}$ in a random potential $V_\mathbf{r}$ with logarithmic correlations, which can also be studied by our RG approach. The same decomposition of disorder $e^{-V_\mathbf{r}} = z_\mathbf{r} e^{-V_\mathbf{r}^>}$, and renormalization via fusion of environments ($z = z' + z''$) yields directly (7) for the distribution $P_l(z)$. Elimination of scales up to $l^* = \ln(L/a)$ yields that $Z \approx z(l^*)$ has the same distribution as the free energy of the DP on the Cayley tree confirming the conjecture of [1,4]. It also yields the multifractal spectrum of $|\psi_0|^2$ found to agree with [1,2]

To conclude, we developed a RG approach to random XY models, disordered CG, and random Dirac problems. By following the whole fugacity distribution, it appears perturbative in the concentration of rare favorable regions, which corresponds to the vector fugacity in the replica method. This expansion is highly nonperturbative in the original fugacity y. A precise connection to the free energy distribution of DP on Cayley trees and random energy models (GREM) arises from the RG [28] and turns out to be crucial to describe the disorder driven transition.

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*Laboratoire propre associé à l'ENS et à Univ. Paris-Sud.

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[17] For $V(\theta)$ defined as $e^{-V(\theta)} = \sum_p e^{-(K/2\pi)(\theta - 2\pi p)^2}$.
[18] As will appear here (2) is equivalent to the random sine Gordon model $H = \int_r (\nabla\phi)^2 + i\mathbf{a}\cdot\nabla\phi + z_+ e^{i\phi} + z_- e^{-i\phi}$ with a natural splitting of disorder $\mathbf{a}, z_\pm$. The replica operator product expansion of $Y_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{\Phi}}$ also yields (4a) and (4b).
[19] Higher charges, e.g., ± 2, are less relevant since the diffusion operator for $\int_{z_\pm} P(z_\pm, z_{++}, z_{--})$ is as in (3) with $K \to 4K$ and $\sigma \to 2\sigma$ and fusion leads to $P(z_{++} \sim 1) \sim P(z_+ \sim 1)^2$.
[20] The fraction of fusioned environments yields the universal factor $\partial_l V/V = 2$.
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[22] $d' = 2\pi^2$, $d = \pi$ for our cutoff (d'/d^2 universal). The cubic term in $\partial_l Y$ in [21] drops out for $m \to 0$.
[23] Equation (3) corresponds to a given branching process, associated with a particular cutoff, which even at $\sigma = 0$ contains disorder in the positions of the branching nodes. The most appropriate cutoff would yield $Y[\mathbf{n}] = y^{\mathbf{n}^2}$ corresponding to a nonlinear term $F[G] = (G - 1)\ln(1 - G)$ in (7).
[24] D. Carpentier and P. Le Doussal (to be published).
[25] This amounts to neglect terms of order $P(1)^3$ in the RG equation for $P(1)$ and in (5) since $P(z_+ \sim 1, z_1 \sim 1)$ consistently remains of order $P(1)^2$.
[26] This is confirmed by a numerical simulation of (6).
[27] Subdominant contributions are neglected, e.g., the variations of D_l.
[28] Direct replica solution [5] of GREM models requires replica symmetry breaking (RSB) for $T < T_g$. It yields exponential free energy distributions (Boltzman weights $z_\mathbf{r}/\sum_{\mathbf{r}'} z_{\mathbf{r}'}$ being dominated by a few states, a characteristic of RSB). Within the RG it translates into generation of broad distributions $P(z) \sim z^{-(1+\mu)}$ ($\mu < 1$, no first moment) which have similar properties. This suggests to follow broad distributions alternatively in a RG with Gaussian distributions but with RSB [24].