Universality in Chiral Random Matrix Theory at $\beta = 1$ and $\beta = 4$

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In this paper the kernel for the spectral correlation functions of invariant chiral random matrix ensembles with real ($\beta = 1$) and quaternion real ($\beta = 4$) matrix elements is expressed in terms of the kernel of the corresponding complex Hermitian random matrix ensembles ($\beta = 2$). Such identities are exact in case of a Gaussian probability distribution and, under certain smoothness assumptions, they are shown to be valid asymptotically for an arbitrary finite polynomial potential. They are proved by means of a construction proposed by Brézin and Neuberger. Universal behavior of the eigenvalues close to zero for all three chiral ensembles then follows from microscopic universality for $\beta = 2$ as shown by Akemann, Damgaard, Magnea, and Nishigaki. [S0031-9007(98)06547-8]

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Since its introduction in nuclear physics [1], random matrix theory (RMT) has been applied successfully to many different branches of physics ranging from atomic physics to quantum gravity (for a recent comprehensive review, we refer to [2]). One important common ingredient is that eigenvalue correlations appear to be insensitive to the details of the underlying Hamiltonian. The success of RMT is based on this type of universality, and it is no surprise that it has received a great deal of attention in recent literature [3–27]. What has been shown is that spectral correlators on the scale of the average eigenvalue spacing are insensitive to the details of the probability distribution of the matrix elements. Because of its mathematical simplicity most studies were performed for complex ($\beta = 2$) Hermitian RMT's. However, in the case of the classical RMT's it was shown that universality extends to real ($\beta = 1$) and quaternion real ($\beta = 4$) matrix ensembles [4,6,7]. This suggests that relations between correlation functions for different values of β which can be derived for a Gaussian probability distribution [28,29] might be valid for a wide class of probability distributions. The main goal of this paper is to establish such general relations. As a consequence, universality for the much simpler complex ensembles implies universality for the real and quaternion real ensembles.

In this Letter we address the question of microscopic universality for the chiral ensembles. These ensembles are relevant for the description of spectral correlations of the QCD Dirac operator. They also appear in theory of universal conductance fluctuations in mesoscopic systems [30,31]. They are characterized by a spectrum that is symmetric about $\lambda = 0$ and have been applied to correlations of eigenvalues close to zero. According to the Banks-Casher formula [32], this part of the spectrum is directly related to the order parameter Σ of the chiral phase transition [$\Sigma = \lim \pi\rho(0)/V$, where V is the volume of space time and $\rho(\lambda) = \sum_k \delta(\lambda - \lambda_k)$]. It is therefore natural to introduce the microscopic limit where the variable $u = \lambda V \Sigma$ is kept fixed for $V \to \infty$. For example, the microscopic spectral density is defined by [33]

$$\rho_S(u) = \lim_{V\to\infty} \frac{1}{V\Sigma} \left\langle \rho\left(\frac{u}{V\Sigma}\right)\right\rangle, \quad (1)$$

where the average is over the distribution of the matrix elements of the Dirac operator. Successful applications of the chiral ensembles to lattice QCD spectra can be found in [29,34–36].

The chiral random matrix ensembles for N_f massless quarks in the sector of topological charge ν are defined by the partition function [33,37]

$$Z^{\beta}_{N_f,\nu} = \int DW \det{}^{N_f}\begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} e^{-n\beta \mathrm{Tr} V(W^\dagger W)}, \quad (2)$$

where W is a $2n \times (2n + \nu)$ matrix. As is the case in QCD, we assume that ν does not exceed $\sqrt{2n}$. The parameter $2n$ is identified as the dimensionless volume of space time. The matrix elements of W are real [$\beta = 1$, chiral orthogonal ensemble (chOE)], complex [$\beta = 2$, chiral unitary ensemble (chUE)], or quaternion real [$\beta = 4$, chiral symplectic ensemble (chSE)]. For technical reasons we consider only finite polynomial potentials $V(x)$. The simplest case is the Gaussian case with $V(x) = \Sigma^2 x$ (also known as the Laguerre ensemble).

It was shown by Akemann *et al.* [23] that, for $\beta = 2$, the microscopic spectral density and the microscopic spectral correlators do not depend on the potential $V(x)$ and are given by the result [38] for the Laguerre ensemble. For $\beta = 2$ all spectral correlators can be obtained from an orthogonal polynomial kernel corresponding to the probability distribution. In their proof the Christoffel-Darboux formula is used to express this kernel in terms of large order polynomials. Microscopic universality then follows from the asymptotics of orthogonal polynomials. As a remarkable achievement, they were able to generalize the relation for Laguerre polynomials,

$$\lim_{n\to\infty} n^{-a} L_n^a\left(\frac{x}{n}\right) = x^{-a/2} J_a(2\sqrt{x}), \quad (3)$$

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to orthogonal polynomials corresponding to an arbitrary polynomial potential. However, their work cannot easily be generalized to $\beta = 1$ and $\beta = 4$. The main result of the present work is a relation between the kernels for the correlation functions of the chOE and chSE and the kernel of the chUE. This relation is exact for the Gaussian ensembles and is valid asymptotically for an arbitrary polynomial potential. For $\beta = 4$, this relation shows universality of the microscopic spectral density and correlators (for $\beta = 1$ only a partial proof was obtained).

The partition function (2) is invariant under $W \to U^\dagger W V$ where the matrices U and V with dimensions determined by W are orthogonal for $\beta = 1$, unitary for $\beta = 2$, and symplectic for $\beta = 4$. This invariance makes it possible to express the partition function (2) in terms of the eigenvalues x_k of $WW^\dagger$ as

$$Z^\beta_{N_f,\nu} = \int \prod_k dx_k\, x_k^{2a} |\Delta(x_i)|^\beta e^{-n\beta \sum_k V(x_k)}, \qquad (4)$$

where the Vandermonde determinant is defined by $\Delta(x_i) = \prod_{k<l}(x_k - x_l)$ and $2a = N_f - 1 + \beta\nu/2 + \beta/2$.

For $\beta = 2$, the spectral correlation functions can be evaluated [39] by expressing the Vandermonde determinant in terms of the orthogonal polynomials defined by

$$\int_0^\infty dx\, e^{-2\phi_a(x)} q_k^{2a}(x) q_l^{2a}(x) = \delta_{kl}\,, \qquad (5)$$

where we have introduced the potential $\phi_a(x) = nV(x) - a\log x$. By using orthogonality relations it can be shown that all spectral correlation functions can be expressed in terms of the kernel

$$K^{2a}_{2n}(x,y) = \sum_{k=0}^{2n-1} q_k^{2a}(x) q_k^{2a}(y)\,. \qquad (6)$$

The spectral density is given by $K^{2a}_{2n}(x,x)\exp[-2\phi_a(x)]$. Microscopic universality then follows from the following generalization of (3) [23]:

$$\lim_{n\to\infty} \sqrt{h_k^{2a}}\, q_k^{2a}\left(\frac{x^2}{n^2}, n\right)\Bigg|_{k=tn} = \Gamma(2a+1)\, \frac{J_{2a}[u(t)x]}{[u(t)x/2]^{2a}}\,, \qquad (7)$$

in the normalization $q_k^{2a}(0)\sqrt{h_k^{2a}} = 1$. The function $u(t)$ follows from the asymptotic properties of the leading order coefficients of the $q_k^{2a}(x)$ and the normalizations h_k^{2a}. Its value at $t = 1$ is given by $u(1) = 2\pi\rho(0)$.

In order to perform the integrations by means of orthogonality relations for $\beta = 1$ and $\beta = 4$, one has to introduce the skew-orthogonal polynomials [39,40]. Below, we first discuss the case $\beta = 1$ and then give general outlines for the case $\beta = 4$.

For $\beta = 1$, the skew orthogonal polynomials of the second kind are defined by

$$\langle R_i, R_j\rangle_R = J_{ij}\,. \qquad (8)$$

with the skew orthogonal scalar product

$$\langle f, g\rangle_R = \int_0^\infty dx\, e^{-2\phi_a(x)} f(x) \hat{Z} g(x)\,, \qquad (9)$$

and nonzero matrix elements of J_{ij} given by $J_{2k,2k+1} = -J_{2k+1,2k} = -1$. The operator $\hat{Z}$ is defined by

$$\hat{Z} g(x) = \int_0^\infty dy\, e^{\phi_a(x)} \epsilon(x-y) e^{-\phi_a(y)} g(y)\,, \qquad (10)$$

Here, $\epsilon(x) = x/2|x|$. It can be shown that all correlation functions can be expressed in terms of the kernel [39,40]

$$K_1(x,y) = \int_0^x dz\, e^{-\phi_a(z)} k_1(y,z) e^{-\phi_a(y)}, \qquad (11)$$

where we have introduced the prekernel

$$k_1(y,z) = \sum_{i,j=0}^{2n-1} R_i(y) J_{ij} R_j(z)\,. \qquad (12)$$

In particular, the spectral density is given by

$$\rho(x) = K_1(x,x) - \frac{1}{2} K_1(\infty,x)\,. \qquad (13)$$

A general scheme for the construction of skew-orthogonal polynomials was introduced by Brézin and Neuberger [41]. The idea is to express them in terms of orthogonal polynomials defined by (5). For technical reasons we expand in the polynomials q_k^{2a+1} (with weight function $x^{2a+1}\exp[-2nV(x)]$). The skew-orthogonal polynomials of degree i can thus be expressed as

$$R_i(x) = \sum_{j=0}^{i} T_{ij} q_j^{2a+1}(x)\,, \qquad (14)$$

where T is a lower triangular matrix with nonvanishing diagonal elements. An essential role is played by the inverse $\hat{L}$ of the operator $\hat{X}^{-1}\hat{Z}$ with $\hat{Z}$ defined in (10) and $\hat{X}g(x) = xg(x)$. It can be easily verified that

$$\hat{L} = \hat{X}[\hat{\partial} - \phi_a'(\hat{X})] + \hat{1}\,. \qquad (15)$$

The matrix representations of the operators $\hat{X}$, $\hat{X}\hat{\partial}$, $\hat{X}^{-1}\hat{Z}$, and $\hat{L}$ in the basis q_k^{2a+1} will be denoted by X_{kl}, D_{kl}, Y_{kl}, and L_{kl}, respectively (with the convention $\hat{X}q_k = X_{kl}q_l$, etc.). In the remainder of this derivation the index $2a+1$ will be suppressed.

In matrix notation (8) can be rewritten as

$$TYT^T = -J\,. \qquad (16)$$

By using that $LY = 1$, this relation can be expressed as

$$L = T^T J T\,. \qquad (17)$$

It can be shown that the matrix L_{kl} is a band matrix with width determined by the order of the polynomial potential $V(x)$. It then follows that T is a band matrix as well [41]. For example, for a Gaussian potential we have that $T_{2m,k} = a_0\delta_{2m,k}$ and $T_{2m+1,k} = b_0\delta_{2m+1,k} + b_1\delta_{2m,k} + b_2\delta_{2m-1,k}$ with coefficients derived in [42,28].

It turns out that we do not need explicit expressions for the T_{ij}. The prekernel (12) can be expressed as

$$k_1(x,y) = \sum_{i,j=0}^{2n-1} \sum_{k \le i} \sum_{l \le j} q_k(x) T_{ki}^T J_{ij} T_{jl} q_l(y). \tag{18}$$

In this relation the indices i and j run up to $2n-1$ in contradistinction to the relations (17) where they run up to ∞. However, it follows from the band structure of L that the number of terms outside the range in (18) is of the same order as the degree of the polynomial potential which is finite. These terms are negligible in the continuum limit of the type (7) where the $q_k^{2a+1}(x)$ and L_{kl} depend smoothly on k and l (notice that L_{kl} is not smooth in $|k-l|$). However, for x around zero and y near the largest zero of $q_l(y)$ we expect potentially non-negligible contributions. We thus have that

$$k_1(x,y) \simeq \sum_{k,l=0}^{2n-1} q_k(x) L_{kl} q_l(y). \tag{19}$$

By means of a partial integration the matrix elements of $\hat{1} - \hat{X}\phi_a'(\hat{X})$ in L can be expressed in terms of the matrix elements of $\hat{X}\hat{\partial}$. This results in

$$L_{kl} = \frac{1}{2}\int_0^\infty z\,dz\, e^{-2\phi_a(z)}[q_l(z) z\partial q_k(z) - q_k(z) z \partial q_l(z)]$$
$$= \frac{1}{2}(D_{kl} - D_{lk}). \tag{20}$$

The matrix elements of D can be reexpressed as $x\partial_x$ or $y\partial_y$. We finally arrive at a remarkably simple expression for $k_1(x,y)$,

$$k_1(x,y) \simeq \frac{1}{2}(y\partial_y - x\partial_x) K_{2n}^{2a+1}(x,y). \tag{21}$$

With the help of the asymptotic properties of the q_k^{2a} (which are the same as for the Laguerre polynomials) this relation can be further simplified to (up to an overall factor determined by the average spectral density)

$$k_1(x,y) \sim \frac{1}{2}(\partial_y - \partial_x) K_{2n}^{2a}(x,y). \tag{22}$$

This is the central result of this paper. It is valid asymptotically both for x and y close to zero (the hard edge of the spectrum) and for x and y in the neighborhood of the largest eigenvalue (the soft edge of the spectrum) where a continuum limit of the orthogonal polynomials q_k^{2a+1} exists. However, as will be argued below, the result (22) is not valid for x near the hard edge and y at the soft edge of the spectrum. This result relates the orthogonal prekernel to the unitary kernel $K_{2n}^{2a}(x,y)$ which has been studied elaborately in the literature [43–45]. The relation (22) is exact for a Gaussian potential in which case it coincides with the result obtained in [10,28,31,46,47].

Universality of the unitary kernel $K_{2n}^{2a}(x,y)$ at the hard edge has been well established [23] for the chiral ensembles, whereas universality at the soft edge was shown in [16,17]. We therefore expect universal behavior of $k_1(x,y)$ in these domains.

Let us finally focus on the spectral density. Using (13) and (22), for a Gaussian potential it can be expressed as

$$\rho(x) \simeq e^{-\phi_a(x)} \int_0^\infty dy\, e^{-\phi_a(y)} \epsilon(x-y) \frac{1}{2}(\partial_y - \partial_x) \times K_{2n}^{2a}(x,y). \tag{23}$$

In the microscopic limit where $n \to \infty$ at fixed $z \equiv xn^2$ the factor $\exp[-2nV(x)] \to 1$ and K_{2n}^{2a} approaches its universal limit. However, in one of the terms contributing to the integral the microscopic limit and the integration cannot be interchanged. It can be shown that there is an additional contribution with x near zero and y near the edge of the spectrum. Naively taking into account this contribution for non-Gaussian potentials leads to a microscopic spectral density that differs from the universal expression. Alternatively, we have established universality of the microscopic spectral density by means of Monte Carlo simulations. Apparently, the assumptions in the derivation of (21) are violated in this case [Eq. (13)]. In the first term contributing to spectral density, $K_1(x,x)$, the microscopic limit and the integral can be interchanged. This establishes universality of $K_1(x,x)$.

In the case of a Gaussian potential the edge contribution can be obtained from the asymptotic expansion of the Laguerre polynomials in this region (an expression in terms of Airy functions). In a future publication, we hope to establish a possible relation with the universal behavior of the q_k^{2a} near the edge of the spectrum [16,17].

The above analysis carries through for the symplectic ensemble. In this case there are no contributions from the soft edge and universality of the microscopic spectral density can be shown rigorously. For $\beta = 4$ [with an additional factor $1/2$ in the exponent of (4)], the correlation functions can be expressed in terms of the kernel

$$k_4(x,y) = \sum_{i,j=0}^{2n-1} Q_i(x) J_{ij} Q_j(y), \tag{24}$$

where the $Q_i(x)$ are skew orthogonal polynomials of the first kind which are defined by the skew-scalar product

$$\langle f,g\rangle = \int_0^\infty \frac{dx}{x} e^{-2\phi_a(x)} f(x)(\hat{L} - \hat{1}) g(x), \tag{25}$$

with the operator $\hat{L}$ defined in (15). In this case we express the $Q_i(x)$ in terms of the polynomials $q_k^{2a-1}(x)$,

$$Q_i(x) = \sum_{k=0}^{i} S_{ik} q_k^{2a-1}(x). \tag{26}$$

The matrix elements of the operators are also in this basis. From the orthogonality relation $\langle Q_i, Q_j\rangle_Q = J_{ij}$ it can be shown that $S(L-1)S^T = -J$ from which we derive $S^T J S = ZX^{-1}$. Again, due to the band structure of L_{kl}, the range of the summations in this relation and in (24) differs by a finite number of terms which can be neglected in the continuum limit. We thus find

$$k_4(x,y) \simeq \sum_{k,l=0}^{2n-1} q_k^{2a-1}(x)\,(\hat{Z}\hat{X}^{-1})_{kl}\,q_l^{2a-1}(y)$$
$$= e^{\phi_a(y)} \int_0^\infty \frac{dz}{z}\, e^{-\phi_a(z)} \epsilon(y-z) K_{2n}^{2a-1}(x,z)\,. \tag{27}$$

Universality of $k_4(x,y)$ thus follows from universality of $K_{2n}^{2a-1}(x,z)$. This relation is exact for a Gaussian potential and reproduces the result found in [29].

In conclusion, we have shown that relations between the kernels for the chOE and chSE and the kernel for chUE are not accidental but follow from an intriguing underlying mathematical structure. Under certain smoothness assumptions these relations are valid asymptotically for an arbitrary polynomial potential. Microscopic universality for $\beta = 4$ and in part for $\beta = 1$ thus follows from universality at $\beta = 2$ at hard edge of the spectrum.

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