Nonlinear Competition between Small and Large Hexagonal Patterns

Mary Silber
Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois 60208

Michael R. E. Proctor
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom
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Recent experiments by Kudrolli, Pier, and Gollub [Physica D (to be published)] on surface waves, parametrically excited by two-frequency forcing, show a transition from a small hexagonal standing wave pattern to a triangular "superlattice" pattern. We show that generically the hexagons and the superlattice wave patterns bifurcate *simultaneously* from the flat surface state as the forcing amplitude is increased, and that the experimentally observed transition can be described by considering a low-dimensional bifurcation problem. A number of predictions come out of this general analysis. [S0031-9007(98)07195-6]

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In recent years there has been considerable interest in pattern formation in spatially extended nonequilibrium systems [1]. These studies have identified universal mechanisms for the formation of certain patterns. For instance, the origin of the ubiquitous hexagonal pattern can be traced to a symmetry-breaking instability of a spatially homogeneous state that occurs in the presence of Euclidean symmetry. In this Letter we use similar ideas to investigate a family of hexagonal and triangular patterns which are born in the same instability that gives rise to simple hexagons, but have structure on two disparate length scales [2,3]. Kudrolli *et al.* [4] call such structures "superlattice patterns"; Fig. 1 gives two examples. Current interest in superlattice patterns is sparked by recent observations of their formation in experiments on parametrically excited surface waves [4] and in nonlinear optics [5], as well as in a study of Turing patterns in reaction-diffusion systems [6].

In this Letter we show that superlattice patterns can arise *directly* from the spatially uniform state via a transcritical bifurcation. Moreover, this occurs in generic pattern-forming systems with only *one* unstable wave number. We show that the triangular superlattice pattern differs from simple patterns such as hexagons and complex quasipatterns, because it is characterized by *both* an amplitude and a *phase* which depend on the distance from the bifurcation point. In order to investigate the phase we find that it is necessary to include high order resonant interaction terms in the amplitude equations. When these terms are neglected the problem becomes degenerate and superlattice patterns such as those in Fig. 1 become just two isolated examples in a continuum of states with varying symmetry. Thus high order terms are essential to understanding superlattice patterns.

While much of our analysis is quite general, our presentation focuses on the recent experiments of [4], in which surface waves are parametrically excited by subjecting a fluid layer to a periodic vertical acceleration involving two rationally related frequencies. In the classic Faraday problem, with single frequency forcing, the first modes to lose stability with increased acceleration are subharmonic with respect to the vibration frequency [7]. With two frequencies, harmonic response can occur [8,9]. The triangular superlattice pattern in Fig. 1 was obtained with an acceleration $f(t) = a[\cos(\chi)\cos(6\omega t) + \sin(\chi)\cos(7\omega t + \phi)]$. In the experiments, a hexagonal standing wave pattern is produced at the onset of instability, then, as the acceleration is increased, there is a transition to the superlattice pattern in Fig. 1 [4]. Comparison of the Fourier transform of the onset hexagons and the superlattice patterns reveals that the new state is formed from the nonlinear interaction of twelve prominent Fourier modes with wave number k_c that lie on a hexagonal lattice with fundamental wave number $k_c/\sqrt{7}$. In this Letter we address the transition between hexagons and superlattice patterns by considering a degenerate bifurcation problem akin to that which explains the transition between hexagons and rolls in many systems. A number of concrete predictions about the form, origin, and stability of the superlattice patterns come out of this symmetry-based analysis.

The experimentally observed patterns are put into a theoretical framework by restricting to solutions that tile the plane in a hexagonal fashion. The time-periodic forcing leads us to a formulation in terms of a stroboscopic map. Since we seek spatially periodic solutions, we express all fields in terms of double Fourier series; for example, the free surface height, at time $t = mT$, is

$$\zeta_m(\mathbf{x}) = \mathrm{Re}\left(\sum_{\mathbf{n}\in\mathbf{Z}^2} w_{\mathbf{n}}(mT)e^{i(n_1\mathbf{k}_1+n_2\mathbf{k}_2)\cdot\mathbf{x}}\right), \qquad (1)$$

where T is the forcing period. We obtain patterns periodic on a hexagonal lattice when $\mathbf{k}_1, \mathbf{k}_2$ satisfy $|\mathbf{k}_1| = |\mathbf{k}_2| = k$ and $\mathbf{k}_1 \cdot \mathbf{k}_2 = -\frac{1}{2}k^2$. A standard *ansatz* in pattern selection problems is to set $k = k_c$, where k_c is the critical wave number of the instability. However, here we want to investigate competition between small and large hexagonal

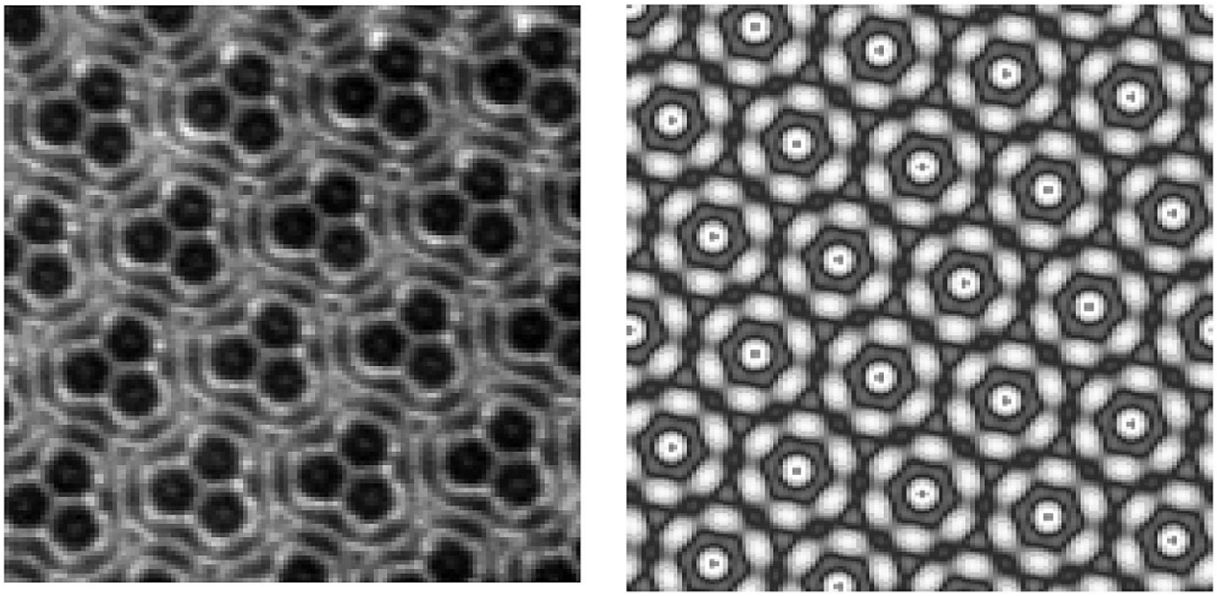

FIG. 1. On the left is a blowup of the experimental superlattice standing wave pattern [4] (courtesy of Kudrolli, Pier, and Gollub). We show a region with area $\sim 1/30$ of the circular container's cross-sectional area. Note that the pattern is periodic on a hexagonal lattice, and that it has triangular symmetry. On the right is a hexagonal superlattice Turing pattern obtained from a numerical integration of a reaction-diffusion system [6] (courtesy of Judd).

patterns, which we do following the approach in [3]. Specifically, we choose k so that twelve wave vectors $\mathbf{K_n} = n_1\mathbf{k_1} + n_2\mathbf{k_2}$ in (1) satisfy $|\mathbf{K_n}| = k_c$, i.e., so there exists a coprime integer pair $\mathbf{n} = (\alpha, \beta)$, $\alpha > \beta > 0$, such that

$$|\alpha\mathbf{k_1} + \beta\mathbf{k_2}| = k\sqrt{\alpha^2 + \beta^2 - \alpha\beta} = k_c\,. \qquad (2)$$

Figure 2 presents an example associated with $\alpha = 3, \beta = 2$, $k_c/k = \sqrt{7}$. The lattice points on the critical circle lie at the vertices of two hexagons rotated by an angle θ relative to each other. The angle, determined by the dot product of $\mathbf{K}_1 = \alpha\mathbf{k}_1 + \beta\mathbf{k}_2$ and $\mathbf{K}_4 = \alpha\mathbf{k}_1 + (\alpha - \beta)\mathbf{k}_2$, satisfies $\cos(\theta) = \frac{\alpha^2 + 2\alpha\beta - 2\beta^2}{2(\alpha^2 - \alpha\beta + \beta^2)}$. If only those modes associated with one of the hexagons are excited, then the pattern periodicity is dictated by k_c and one recovers the standard six-dimensional hexagonal lattice bifurcation problem. However, if all twelve modes are excited, then the period of the pattern is greater by a factor of k_c/k. In order to simplify our presentation we set $\alpha = 3, \beta = 2$; the results for general α, β are similar.

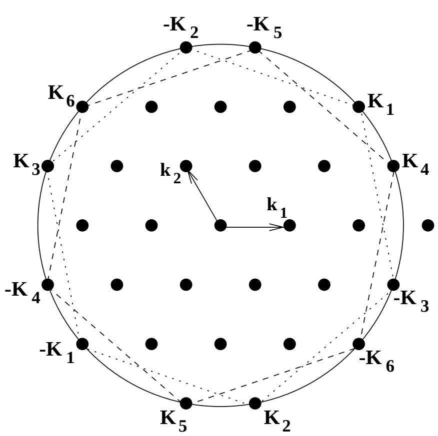

FIG. 2. Intersection of the critical circle $|\mathbf{k}| = k_c$ with the $\mathbf{k}$-space hexagonal lattice generated by $\mathbf{k}_1$, $\mathbf{k}_2$. In this example $\alpha = 3, \beta = 2$ in (2). The wave vectors $\pm\mathbf{K}_1, \ldots, \pm\mathbf{K}_3$ lie at the vertices of a hexagon, as do $\pm\mathbf{K}_4, \ldots, \pm\mathbf{K}_6$.

In formulating the bifurcation problem, we assume that the flat fluid surface loses stability to harmonic waves of wave number k_c as a bifurcation parameter λ is increased through zero. Thus the Faraday instability sets in when a Floquet multiplier μ crosses the unit circle at $\mu = 1$. In our formulation there are twelve Fourier modes in (1) that are neutrally stable at $\lambda = 0$; all others are damped. In this case the map can be reduced, near the onset of instability, to a twelve-dimensional center manifold. Let $z_j(m)$ be the complex amplitude, at $t = mT$, of the Fourier mode $e^{i\mathbf{K}_j\cdot\mathbf{x}}$, with $\bar{z}_j$ the amplitude of $e^{-i\mathbf{K}_j\cdot\mathbf{x}}$. Here the $\mathbf{K}_j$ are labeled as in Fig. 2.

The possible nonlinear terms in the stroboscopic map $\mathbf{z}(m + 1) = \mathbf{f}(\mathbf{z}(m))$, $\mathbf{z} \equiv (z_1, \ldots, z_6)$, are restricted by the symmetries of the problem. The first component of $\mathbf{f}$ has the general form

$$f_1 = h_1(\mathbf{u}, \mathbf{q}, \bar{\mathbf{q}})z_1 + h_2(\mathbf{u}, \mathbf{q}, \bar{\mathbf{q}})\bar{z}_2\bar{z}_3 + c_1 z_3^2 z_4^2 \bar{z}_6 + c_2 \bar{z}_1 z_2 z_4 \bar{z}_5^2 + \mathcal{O}(|\mathbf{z}|^6)\,, \qquad (3)$$

where $\mathbf{u} \equiv (|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2, |z_5|^2, |z_6|^2)$, and $\mathbf{q} \equiv (z_1z_2z_3, z_4z_5z_6)$. The discrete hexagonal symmetries place further restrictions on the functions h_1, h_2 and determine the other components of $\mathbf{f}$ from f_1 [3]. We note that if the resonant interaction terms, with coefficients c_1 and c_2, are neglected, then the phases ϕ_j associated with each $z_j \equiv R_j e^{i\phi_j}$ enter the problem only through the total phases $\Phi_1 \equiv \phi_1 + \phi_2 + \phi_3$ and $\Phi_2 \equiv \phi_4 + \phi_5 + \phi_6$. As a result the cubic truncation of (3) admits solutions in the form of two hexagons rotated relative to each other by $\theta \approx 22°$ (for $\alpha = 3, \beta = 2$), and translated relative to each other by *arbitrary* amounts. Kudrolli *et al.* [4] noted that only a restricted set of relative translations of the two hexagons leads to a superlattice pattern with triangular symmetry. Their phenomenological description did not, however, give a mechanism for selecting a particular shift. We now investigate an alternative description of the triangular superlattice pattern which has the advantage that it is, in fact, consistent with the inclusion of the high order resonant interaction terms.

We focus on patterns that have at least threefold rotational symmetry by restricting the problem to the subspace $\mathbf{z}(m) = (u_m, u_m, u_m, v_m, v_m, v_m)$, where u_m, v_m are complex. First we recall some results about period-one simple hexagons which satisfy $(u_m, v_m) = (Re^{i\varphi}, 0)$ or $(0, Re^{i\varphi})$. The amplitude R and phase φ obey

$$0 = \lambda + \epsilon R\cos(3\varphi) + AR^2 + \ldots,$$
$$0 = \sin(3\varphi)(-\epsilon + BR^2 + \ldots), \qquad (4)$$

where ϵ, A, and B are nonlinear coefficients that arise from the Taylor expansions of h_1 and h_2 in (3). The solutions of (4) that bifurcate from the origin at $\lambda = 0$ satisfy $\sin(3\varphi) = 0$; the hexagons with $\varphi = 0, \pm\frac{2\pi}{3}$ are related by translations, as are the hexagons with $\varphi = \pi, \pm\frac{\pi}{3}$. These two sets of hexagons, "up-hexagons" (H^+) and "down-hexagons" (H^-), form the two branches associated with a transcritical bifurcation. In non-Boussinesq convection, these states correspond to ones in which fluid is rising or falling at the centers of convection cells [10].

Next we consider patterns involving all six modes $z_1, \ldots, z_6$, with $z_j = \rho e^{i\psi}$. The amplitude and phase satisfy, cf. (4),

$$0 = \lambda + \epsilon\rho\cos(3\psi) + \tilde{A}\rho^2 + \ldots,$$
$$0 = \sin(3\psi)(-\epsilon + \tilde{B}\rho^2 + \ldots)$$
$$+ (c_1 - c_2)\rho^3\sin(2\psi) + \ldots. \qquad (5)$$

In this case there are two distinct types of solutions that bifurcate from the origin at $\lambda = 0$, those with phase $\psi = 0, \pi$ and those with $\psi \approx \pm\pi/3, \pm 2\pi/3$. The phase associated with the latter solutions depends on the amplitude, which in turn depends on the distance from the bifurcation. The solutions with $\psi = 0, \pi$ have hexagonal symmetry; in [3] they are referred to as "superhexagons." As with simple hexagons, superhexagons bifurcate transcritically with the two parts of the branch satisfying $\psi = 0$ and $\psi = \pi$, respectively. The solutions satisfying $\psi \approx \pm\pi/3, \pm 2\pi/3$ have only triangular symmetry, and are new. The triangular solutions with $\psi \approx 2\pi/3$ and $\psi \approx -\pi/3$ form the two parts of a transcritical branch; a rotation of these patterns by π changes the sign of ψ. These solutions have structure identical to the superlattice state described in [4]: compare the triangular superlattice pattern in Fig. 1 with that in Fig. 3. Moreover, we know that solutions with $\cos(3\varphi) > 0$ in (4) and $\cos(3\psi) > 0$ in (5) bifurcate in the same direction from the origin, as do branches with $\cos(3\varphi), \cos(3\psi) < 0$.

Finally, we address the experimentally observed transition between the patterns in Fig. 3. Since all of the primary solution branches are unstable at bifurcation, due to the quadratic term in (3), we assume that the coefficient ϵ of this term satisfies $|\epsilon| \ll 1$. This allows us to investigate stable small amplitude solutions and transitions between them. If we truncate (3) at cubic order, thereby neglecting the high-order resonant terms, then the superlattice patterns are at best neutrally stable. Specifically, while we expect a (multiplicity 2) unit multiplier associated with translations, the extra multiplier (of multiplicity 2) results from a symmetry of the *truncated* equations that allows a *relative* translation of the two rotated hexagons that make up the pattern. As noted above, this symmetry is broken when the resonant terms are included; the Floquet multiplier μ then moves off the unit circle. In particular, $\mu > 1$ for the superhexagons if $4c_1 + 5c_2 < 0$ [3]. Analogously, we can show that $\mu < 1$ for the triangular superlattice pattern if $4c_1 + 5c_2 < 0$. Thus if the superlattice patterns are *neutrally* stable when the high-order resonant terms are neglected, then we expect that one of them is stable and the other is, in fact, unstable. This is consistent with the experimental observations of Kudrolli *et al.* [4] who only observe the triangular superlattice pattern.

In Fig. 4, we present part of a bifurcation diagram associated with the map. Specifically, we keep all terms through cubic and the essential quintic terms:

$$f_1 = (1 + \lambda)z_1 + \epsilon\bar{z}_2\bar{z}_3$$
$$+ (a_1|z_1|^2 + a_2|z_2|^2 + a_2|z_3|^2)z_1$$
$$+ (a_4|z_4|^2 + a_5|z_5|^2 + a_6|z_6|^2)z_1$$
$$+ c_1 z_3^2 z_4^2 \bar{z}_6 + c_2\bar{z}_1 z_2 z_4 \bar{z}_5^2, \qquad (6)$$

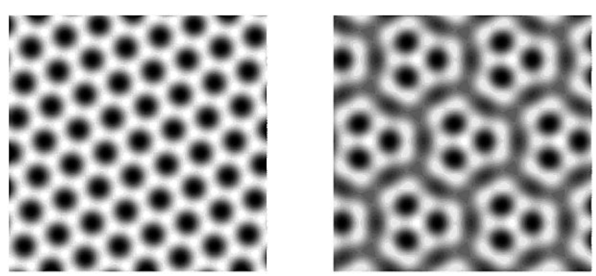

FIG. 3. Examples of patterns, periodic on a hexagonal lattice, that bifurcate transcritically from the flat state at $\lambda = 0$. These are plots of appropriate superpositions of critical Fourier modes with wave vectors $\pm\mathbf{K}_1, \ldots, \pm\mathbf{K}_6$ of Fig. 2. The plots are, left to right, down-hexagons (H^-) and superlattice down-triangles (T^-) (cf. Fig. 1). The critical wave number k_c dictates the size of the small scale structure evident in the superlattice pattern.

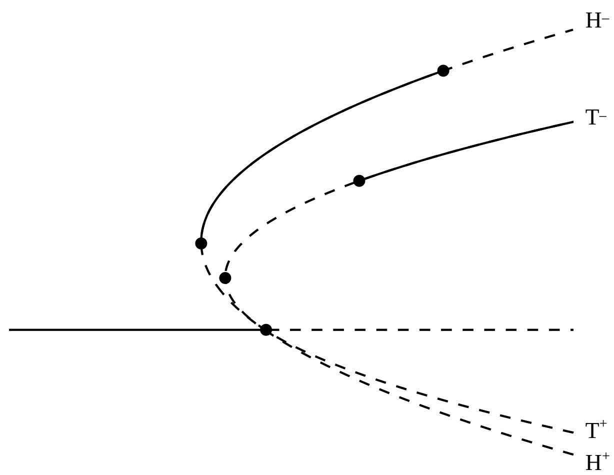

FIG. 4. Part of the bifurcation diagram associated with the bifurcation problem (6) for $-1 \ll \epsilon < 0$. Bifurcation points are indicated by solid circles. Only branches of simple hexagons ($H^{\pm}$) and superlattice triangles ($T^{\pm}$) are indicated; solid lines correspond to stable solutions. Branches produced in secondary bifurcations and *unstable* primary branches are not shown.

and assume that the cubic coefficients satisfy

$$a_1 + 2a_2 < -|a_4 + a_5 + a_6|,$$
$$a_2 - a_1 > \sqrt{\frac{(a_4 - a_5)^2 + (a_4 - a_6)^2 + (a_5 - a_6)^2}{2}}, \tag{7}$$

with $4c_1 + 5c_2 < 0$. Conditions (7) ensure that superlattice patterns eventually supersede the simple hexagons, and that the stripe pattern is never stabilized.

This example demonstrates that it is possible to reproduce the type of transition reported in [4] within the framework of a generic bifurcation problem. Moreover, a number of predictions come out of this analysis. Specifically, we find that *both* the transition between the flat state and the small hexagons and the transition between the hexagons and the superlattice pattern are hysteretic, with the region of bistability between hexagons and the superlattice pattern exceeding that between hexagons and the flat state. While there is no clear detection of hysteresis in the experiments, the transition from hexagons to the superlattice pattern is reported to occur via domains of the superlattice pattern growing, with increased acceleration, until they fill the container [4]. Coexisting domains of different patterns is a common manifestation of bistability of patterns in extended systems. Another prediction that comes out of our analysis is that for $\epsilon < 0$ the transition is between the down states depicted in Fig. 3, while if $\epsilon > 0$ the transition is between up states. While in Rayleigh-Bénard convection there is a clear distinction between up and down states; this distinction may be a subtle one in the Faraday experiment since standard imaging techniques rely on reflection of light from the surface of the fluid, so that surface peaks are not distinguishable from troughs. The nonlinearity inherent in the imaging may also present challenges to extracting the amplitude-dependent phase that we associate with the triangular superlattice patterns.

While the foregoing analysis shows how it is possible to achieve the experimentally observed transitions within a general framework, many of the specific predictions of the analysis still need to be tested. It would be interesting to calculate the nonlinear coefficients in (3) from the hydrodynamic equations to determine whether the minimal inequalities (7) are satisfied for the experimental parameters. Such computations are quite involved in the case of two-frequency forcing of viscous fluids. However, much progress has been made in the case of very low viscosity fluids by Zhang and Viñals [11].

Finally, we note that all of our calculations have assumed that solutions are strictly periodic on a hexagonal lattice. This is certainly an appropriate model for observations of superlattice patterns, but as noted in [4] quasipatterns are observed for other experimental parameters. A mechanism for favoring quasipatterns by a resonant interaction between two bifurcating states with different horizontal wave numbers was proposed by Edwards and Fauve [8], and investigated for a Swift-Hohenberg type of model by Lifshitz and Petrich [12]. Whether a similar resonance mechanism is responsible for the spatially periodic superlattice patterns is an intriguing, open question.

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