Adaptive Frequency Model for Phase-Frequency Synchronization in Large Populations of Globally Coupled Nonlinear Oscillators

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Phase models describing self-synchronization phenomena in populations of globally coupled oscillators are generalized including "inertial" effects. This entails that the oscillator frequencies also vary in time along with their phases. The model can be described by a large set of Langevin equations when noise effects are also included. Also, a description of such systems can be given in the thermodynamic limit of infinitely many oscillators via a suitable Fokker-Planck-type equation. Numerical simulations confirm that simultaneous synchronization of phases and frequencies is possible when the coupling strength goes to infinity. [S0031-9007(98)07062-8]

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A number of phase models have been proposed over the recent years to describe the dynamic behavior of large populations of nonlinear oscillators subject to a variety of coupling mechanisms. A major phenomenon that can be observed is the possibility of self-synchronization among the members of the population. These can represent fireflies, pancreatic beta cells, heart pacemaker cells, and neurons [1,2], as well as circuit arrays and other things (see [3,4] for further references). Such models concern populations of $N \gg 1$ as well as of infinitely many members, and noise terms accounting for random imperfections may also be included. However, doubt in the possibility of effectively synchronizing an entire population of oscillators in practice, both in phase and frequency, has been cast by a recently found "uncertainty principle," in the mean-field coupling model (the Kuramoto-Sakaguchi model [5,6]). Indeed, it was shown in [7] that the Kuramoto-Sakaguchi model with noise terms does *not* allow for simultaneous synchronization in both phase and frequency.

In a more recent paper, Ermentrout [1] revisited the special problem of self-synchronization in populations of fireflies of a certain kind (*the Pteroptyx malaccae*). The Kuramoto-Sakaguchi model yields a too fast approach to the synchronized state (compared to the observed behavior), and also requires an infinite value of the coupling parameter to achieve full phase synchronization. Therefore, Ermentrout proposed, rather, an adaptive frequency model in terms of $N \gg 1$ nonlinearly coupled *second-order* differential equations for the phases, which can handle both problems. Such a model differs from the Kuramoto-Sakaguchi formulation in that the natural frequency of each oscillator is allowed to vary in time, thus leading to a new set of model equations.

It may be of some interest to stress that also certain aftereffects in alterations of circadian cycles in mammalians may be explained by Ermentrout-type models; cf. [1]. Other applications have also been pointed out, for instance, to power systems described by the swing equations [8], and also to extend the analysis of certain Hamiltonian systems [9]. Moreover, several instances of Josephson junctions arrays have been described in simplified versions [10,11], where nonlinear first-order phase equations govern the dynamics of zero temperature circuits. Nonzero temperature effects could be included, however, adding suitable noise terms, and second-order time derivatives might yield a physically more satisfactory picture.

Tanaka *et al.* [12], on the other hand, considered about the same problem described by Ermentrout, but within the mean-field coupling framework and with sinusoidal nonlinearities. In the light of a kind of uncertainty principle [7], which governs phase-frequency synchronization processes in the Kuramoto-Sakaguchi models, here we extend the Ermentrout-Tanaka *et al.* analysis proposing a new model. This consists of a system of $N \gg 1$ (but $N < \infty$) second-order Langevin equations subject to a mean-field interaction with a sinusoidal nonlinearity. Also, in the thermodynamic limit $N \to \infty$, we propose a certain nonlinear partial integro-differential (Fokker-Planck-type) equation. The latter yields the time evolution of the one-oscillator probability density of the system. As a justification of our assumptions, we stress that the sinusoidal nonlinearity can indeed be representative of more general types of nonlinearities as long as the natural frequencies fall within the range of the adaptive frequency [1,12]. On the other hand, a mean-field model can be adopted as a reasonable one, as pointed out by Ermentrout [1], as long as we are concerned with rather compact populations of fireflies lying on nearby trees. Also, in case of Josephson junctions arrays [11], the all-to-all coupling (corresponding to the mean-field model) can indeed be justified by circuit analysis, rather than because of a merely simplifying approximation. In addition, however, here we introduce some noise terms, so as to account for unavoidable imperfections of various natures. Therefore, the model we propose is given by

$$m\,\ddot{\theta}_j + \dot{\theta}_j = \Omega_j + K\,r_N \sin(\psi_N - \theta_j) + \xi_j(t)\,, \qquad j = 1,\ldots,N\,, \quad (1)$$

or by the system

$$\dot{\theta}_j = \omega_j\,,$$
$$\dot{\omega}_j = \frac{1}{m}[-\omega_j + \Omega_j + K\,r_N \sin(\psi_N - \theta_j)] + \frac{1}{m}\,\xi_j(t), \qquad j = 1,\ldots,N\,, \quad (2)$$

where $\theta_j, \omega_j, \Omega_j$ denote phases, frequencies, and natural frequencies, $m > 0$ is an "inertial term," and K sizes the nonlinearity. The complex order-parameter, defined by

$$r_N e^{i\,\psi_N} = \frac{1}{N}\sum_{j=1}^{N} e^{i\,\theta_j}, \quad (3)$$

measures the phase synchronization, and the ξ_j's are Gaussian white noises, with $\langle \xi_j \rangle = 0$, $\langle \xi_i(t)\xi_j(s)\rangle = 2D\,\delta_{ij}\delta(t - s)$.

Typically, N must be large, but we are also interested in the limit of infinitely many oscillators. In this case we obtain, for the one-oscillator probability density, $\rho(\theta, \omega, \Omega, t)$, the evolution equation

$$\frac{\partial \rho}{\partial t} = \frac{D}{m^2}\frac{\partial^2 \rho}{\partial \omega^2} - \frac{1}{m}\frac{\partial}{\partial \omega}[(-\omega + \Omega + Kr\sin(\psi - \theta))\rho] - \omega\,\frac{\partial \rho}{\partial \theta}\,, \quad (4)$$

which should be accompanied by initial and boundary data (2π periodicity in θ, and decay to zero as $\omega \to \pm\infty$, with sufficiently high rate), and normalization, $\int_{-\infty}^{+\infty}\int_0^{2\pi} \rho(\theta, \omega, \Omega, 0)d\omega\, d\theta = 1$.

In Eq. (4), r and ψ are given by

$$re^{i\psi} = \int_{-\infty}^{+\infty} d\omega \int_0^{2\pi} d\theta \times \int_{-\infty}^{+\infty} d\Omega\; g(\Omega) e^{i\theta} \rho(\theta, \omega, \Omega, t)\,, \quad (5)$$

that is, the complex *phase* order-parameter, whose amplitude measures the degree of the phase synchronization. In (5), $g(\Omega)$ represents a given natural frequency distribution. In order to study, in cases of both finitely and infinitely many oscillators, simultaneous self-synchronization in phase *and* frequency, it is convenient to introduce in addition, as in [7], the complex *frequency* order-parameter,

$$s_N e^{i\,\phi_N} = \frac{1}{N}\sum_{j=1}^{N} e^{i\,\omega_j} \qquad (N < \infty)\,, \quad (6)$$

$$s\,e^{i\phi} = \int_{-\infty}^{+\infty} d\omega \int_0^{2\pi} d\theta \times \int_{-\infty}^{+\infty} d\Omega\; g(\Omega) e^{i\omega} \rho(\theta, \omega, \Omega, t) \qquad (N = \infty)\,. \quad (7)$$

In the following, we take for simplicity identical oscillators, $g(\Omega) = \delta(\Omega)$. In order to analyze the spread in phase and frequency, we solve the stationary equation associated with (4). To this purpose, we look for solutions of the form $\rho(\theta, \omega) = \chi(\theta)\eta(\omega)$. Thus,

$$\left(\frac{D}{m^2}\frac{d^2\eta}{d\omega^2} + \frac{1}{m}\,\omega\,\frac{d\eta}{d\omega} + \frac{1}{m}\,\eta\right)\chi - \frac{1}{m}\,Kr\sin(\psi - \theta)\chi\,\frac{d\eta}{d\omega} - \omega\,\eta\,\frac{d\chi}{d\theta} = 0\,. \quad (8)$$

Numerical simulations show that the frequency distribution $\eta(\omega, t) = \int_0^{2\pi} \rho(\theta, \omega, t)d\theta$ does *not* seem to depend on the coupling strength K; cf. the time evolution of the frequency order-parameter, $|s(t)|$, Fig. 1.

Therefore, looking for solutions $\eta(\omega)$ independent of K, we obtain from Eq. (8)

$$\frac{D}{m^2}\frac{d^2\eta}{d\omega^2} + \frac{1}{m}\,\omega\,\frac{d\eta}{d\omega} + \frac{1}{m}\,\eta = 0\,,$$
$$-\frac{1}{m}\,Kr\sin(\psi - \theta)\chi\,\frac{d\eta}{d\omega} - \omega\,\eta\,\frac{d\chi}{d\theta} = 0\,. \quad (9)$$

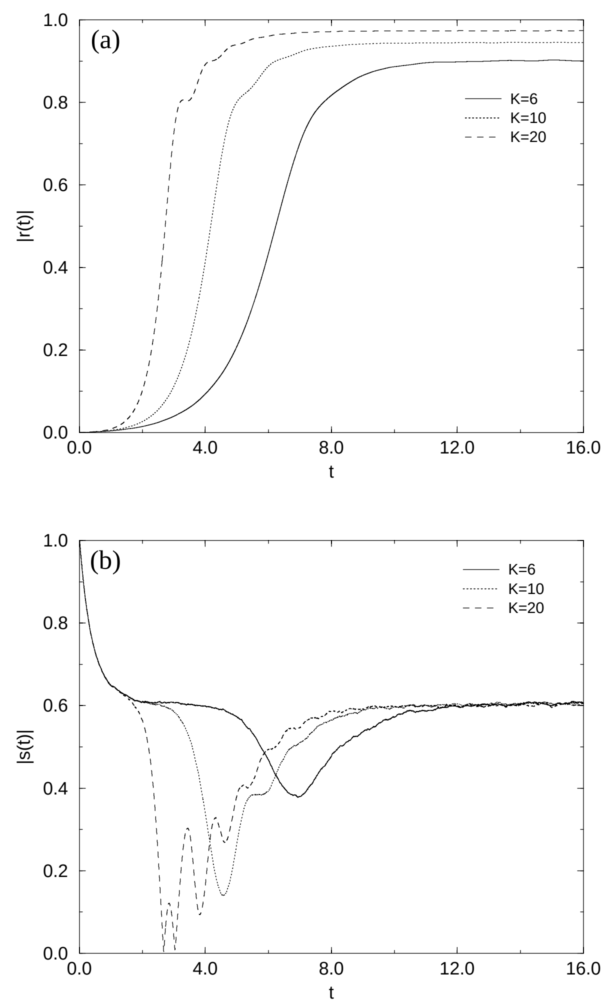

FIG. 1. Time evolution of the order-parameter amplitudes, $|r(t)|$ (a) and $|s(t)|$ (b). The parameter m is kept fixed to 1, the coupling strength is $K = 6$ (solid line), $K = 10$ (dotted line), and $K = 20$ (dashed line), and $D = 1$.

Using the normalization conditions and the boundary condition, $\eta(\omega) \to 0$ as $\omega \to \pm\infty$, the solution to Eq. (9) is promptly obtained:

$$\eta(\omega) = \sqrt{\frac{m}{2\pi D}}\, e^{-(m/2D)\omega^2},$$
$$\chi(\theta) = \frac{e^{(K/D)\, r \cos(\psi-\theta)}}{\int_0^{2\pi} e^{(K/D)\, r \cos(\psi-\theta)} d\theta}. \tag{10}$$

Define the spread in phase and frequency as

$$(\Delta\theta)^2 = \langle(\theta - \psi)^2\rangle - (\langle\theta - \psi\rangle)^2,$$
$$(\Delta\omega)^2 = \langle\omega^2\rangle - (\langle\omega\rangle)^2, \tag{11}$$

brackets denoting average with respect to the density distribution ρ. The symmetry properties of the stationary solution to Eq. (4) can be exploited along with the Laplace method to obtain, in the limit of large coupling $K \to \infty$,

$$(\Delta\theta)^2 = \frac{\sqrt{2}\,D}{K}, \qquad (\Delta\omega)^2 = \frac{D}{m}; \tag{12}$$

cf. [7], and, from these, the "uncertainty relation"

$$\Delta\theta\Delta\omega = \frac{2^{1/4} m^{-1/2} D}{K} \tag{13}$$

is obtained immediately.

Numerical simulations of the Monte Carlo type for a large number of oscillators ($N = 30\,000$) were carried out in the system of Langevin equations (2). The distribution function and the amplitudes of both the order-parameters, $r(t)$ and $s(t)$, have been computed for different values of the parameters m, D, K, when the natural frequency distribution is $g(\Omega) = \delta(\Omega)$. Note, in particular, in Fig. 1 that (partial) synchronization in phase is achieved faster and better for larger values of K, while synchronization in frequency remains always constant, as we expected. In Fig. 2, however, (partial) *phase* synchronization is observed (for a fixed value of K), to be independent of m, while the *frequency* synchronization decreases as m gets smaller. This fact can also be observed in Fig. 3, where the phase and frequency distributions are shown.

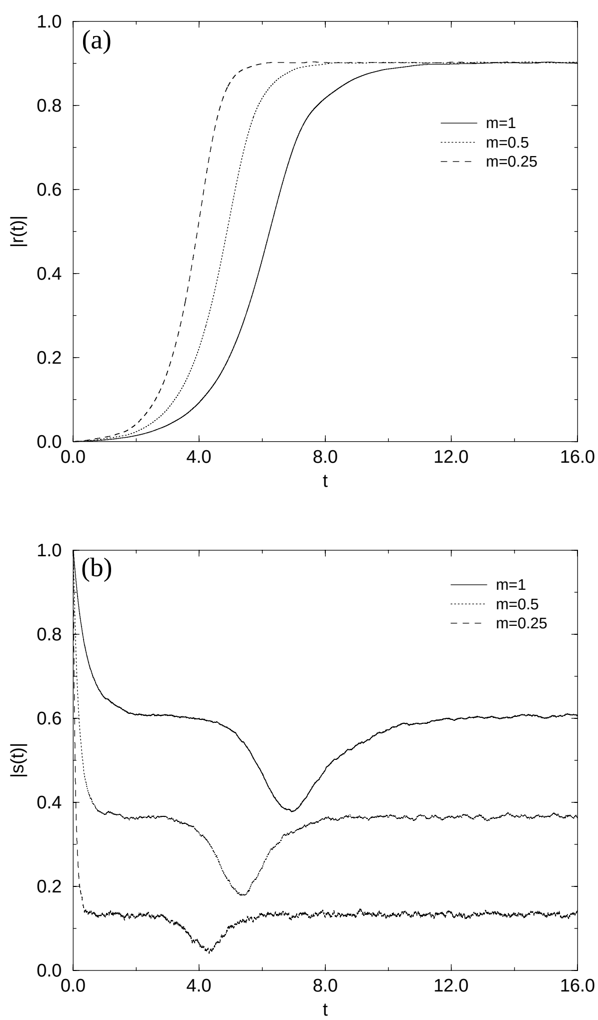

FIG. 2. Time evolution of the order-parameter amplitudes, $|r(t)|$ (a) and $|s(t)|$ (b), for three different values of m. The coupling strength is kept fixed to $K = 6$, and $D = 1$.

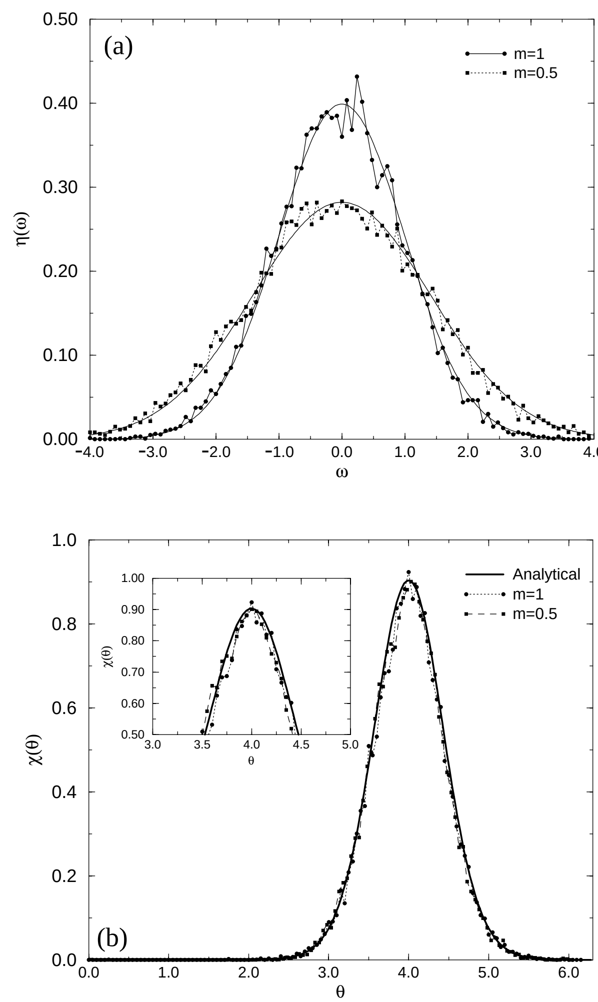

FIG. 3. Frequency (a) and phase (b) distributions for two different values of m. Comparison between the analytical and the numerical solutions is shown. The coupling strength is kept fixed to $K = 6$, and $D = 1$. Details are shown in the inset.

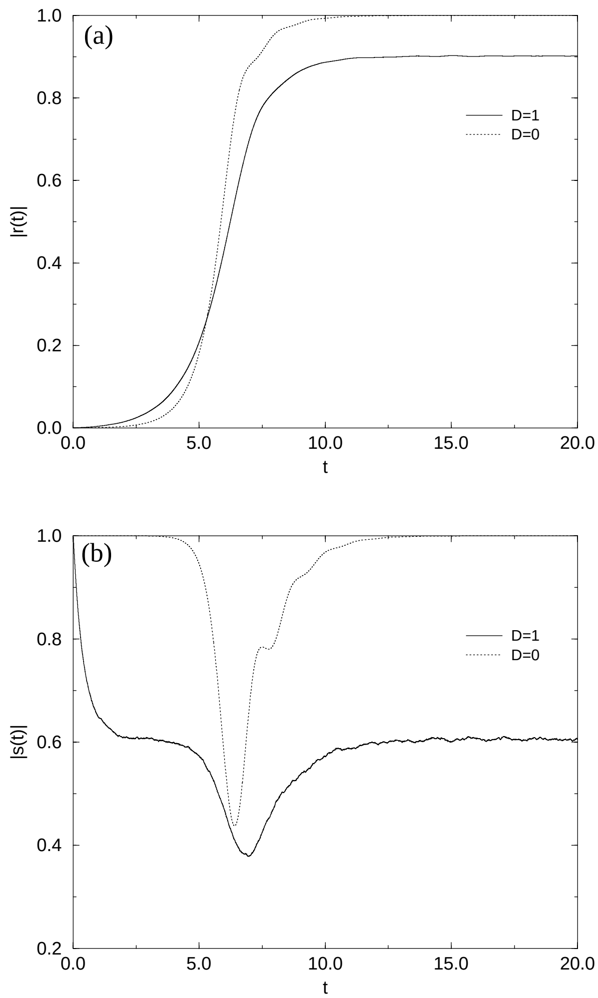

FIG. 4. Time evolution of the order-parameter amplitudes, $|r(t)|$ (a) and $|s(t)|$ (b) for two different values of D. The coupling strength is kept fixed to $K = 6$, and $m = 1$.

In Fig. 4, it is shown that the noise reduces the synchronization in both phase and frequency, as we expected from analytical considerations. Setting (formally) $m = 0$ exactly, in Eq. (1), we recover the Kuramoto-Sakaguchi model with noise, described in the limit of infinitely many oscillators by a Fokker-Planck-type equation for the distribution $\rho(\theta, \Omega, t)$. In this case, the frequency ω, called "drift velocity," arises naturally in the problem as a dependent variable, and is given by

$$\omega = \Omega + Kr \sin(\psi - \theta). \tag{14}$$

The frequency distribution can be obtained from the phase distribution [7], and the spread of both, phase and frequency distributions, becomes

$$(\Delta\theta)^2 = \frac{\sqrt{2}\,D}{K}, \qquad (\Delta\omega)^2 = \sqrt{2}\,DK, \tag{15}$$

and, consequently, the uncertainty relation is

$$\Delta\theta\Delta\omega = \sqrt{2}\,D. \tag{16}$$

It seemed natural to add a noise term in Eq. (1), which corresponds to add a noise independent of the inertial parameter m. One may consider, however, the possibility to introduce such a noise term directly in Eq. (2), thus scaling its effects in a rather different way. It may be interesting to see that the ensuing results are as follows: Eqs. (12) and (13) become now $(\Delta\theta)^2 = \sqrt{2}\,m^2D/K$, $(\Delta\omega)^2 = mD$, and $\Delta\theta\Delta\omega = 2^{1/4}m^{3/2}D/K$, since D has to be replaced by m^2D. The main difference is that $\Delta\theta \to 0$, $\Delta\omega \to 0$, as $m \to 0$, and hence the spread of both phase and frequency, and thus the uncertainty, vanish for vanishing m's. All of this is in (qualitative) agreement with what happens in the Kuramoto-Sakaguchi model for vanishing noise [cf. Eq. (15), and Eq. (1) with $m\xi_j(t)$ replacing $\xi_j(t)$].

In summary, we stress that a *new* model to better explain synchronization phenomena in populations of fireflies has been formulated. It has been emphasized in [1] that the same type of models should also yield an improved picture for the interaction among neurons, merely changing the time and space scales with respect to the fireflies problem. The main feature of the present model seems to be, however, that no uncertainty occurs in synchronizing both phases and frequencies in the limit of infinite coupling strength; cf. Eq. (13) with Eq. (16).

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