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Transitions in Quantum Networks

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We consider transitions in quantum networks analogous to those in the two-dimensional Ising model. We show that for a certain network configuration of active components the transition is between the quantum and the classical behavior of the network, and the critical amplification coincides with the fundamental quantum optical cloning limit. [S0031-9007(98)07112-9]

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Precise control over single quantum systems is essential in testing and harnessing of quantum mechanics. This has become possible with the advances in laser cooling and trapping techniques and manipulation of optical elements in the one-photon level. The availability of single quantum systems has fed the interest in *quantum networks*: A quantum computer [1] is a network of individual quantum systems, where any two of the nodes can interact with each other. Most quantum private communication schemes [2] are networks of two or three nodes. In addition to these information processing and communication related applications, networks of optical components [3] and avoided crossings in multilevel systems [4,5] have been considered in order to study higher dimensional quantum interference effects.

According to statistical physics, a set of probabilistically behaving individual systems can exhibit critical behavior when connected. In this paper we consider the question whether transition phenomena exist in networks of systems which behave probabilistically *not because of finite temperature but due to their quantum nature*; transitions are known to exist in Ising quantum chain models. We define a model of a quantum network which carries in its structure a formal analogy to the two-dimensional Ising model. Such networks can be experimentally realized by various active (energy-consuming) or passive (energy-preserving) components. It is found that transitions do take place and we are able to give them a clear physical interpretation. For active systems the transition

is related to quantum and classical, for passive systems to diabatic and adiabatic behavior of the network. The transition phenomenon is clearly reflected in observable quantities and shows a relation to symmetry which can be of general significance for quantum networks.

The Ising model describes a set of two-state systems which interact with their nearest neighbors; a quantum analogy of such a setup can be experimentally realized in various ways, as will be explained below. Figure 1 shows schematically a 2D quantum network with nearest

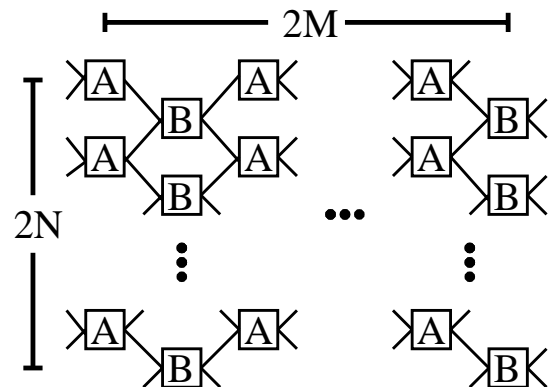


FIG. 1. A quantum network where the nodes are connected to their nearest neighbor. The boxes \boxed{A} and \boxed{B} denote the transformations performed at the nodes; when they are defined by Eq. (1) the network has a relation to the Ising model.

$$P = (F_N \otimes I_2) \begin{bmatrix} \boxed{K_0} & 0_2 & \cdots \\ 0_2 & \ddots & \\ \vdots & & \boxed{K_{N-1}} \end{bmatrix} (F_N^\dagger \otimes I_2), \quad (2)$$

where $(K_n)_{11} = (K_n)_{22}^* = C(\theta)C(\phi) + S(\theta)S(\phi)\omega^{-n}$, $(K_n)_{12} = (K_n)_{21}^* = -C(\phi)S(\theta) + C(\theta)S(\phi)\omega^{-n}$, and $C \equiv \cosh$ and $S \equiv i \sinh$. Most textbooks present the solution of the Ising model in a slightly different form, but we have formulated the problem as in (2) in order to make a connection to interferometers. The usual Mach-Zehnder interferometer affects the input states by a unitary transformation U_{M-Z} which can be formally written as

$$U_{M-Z} = (F_2 \otimes I_1) \begin{bmatrix} K_0 & 0 \\ 0 & K_1 \end{bmatrix} (F_2^\dagger \otimes I_1), \quad (3)$$

where $K_n = \exp(i\phi n)$ is determined by a chosen phase ϕ . Thus the Ising-type network we consider acts like an N -dimensional interferometer where, instead of one-mode phase shifts, two-mode rotations are performed in between the N -dimensional mixers F_N and F_N^\dagger [11].

From K_n one obtains the eigenvalues $e^{\pm\gamma_n}$, where γ_n are determined via

$$\begin{aligned} \cosh \gamma_n &= \cosh \theta \cosh \phi - \cos(2\pi n/N) \sinh \theta \sinh \phi \\ &= \coth \phi \cosh \phi - \cos(2\pi n/N). \end{aligned} \quad (4)$$

An explicit expression for γ_n is given via the integral representation [6] $\gamma_n = \int_0^\pi d\nu/\pi \log[2(\coth \phi \cosh \phi - \cos(2\pi n/N) - \cos \nu)]$. As can be seen from above, the zeroth eigenvalue ($n = 0$) is not a smooth function of ϕ : at $\gamma_0 = 0$, i.e., when $\cosh \phi = \sqrt{2}$, its derivative $\frac{d\gamma_0}{d\phi}$ has a discontinuity. In the Ising model this gives the transition temperature $kT_c = 2.269\epsilon$. We are now at the point to interpret what this mathematical behavior means physically in the case of quantum networks. We will first consider active SU(1,1) networks, then the passive SU(2) ones.

For the active SU(1,1) networks the amplification G of the single components A and B is $\cosh^2 \theta$ and $\cosh^2 \phi$, respectively. The critical amplification $G_c = \cosh^2 \theta_c = \cosh^2 \phi_c = 2$ has an interesting physical interpretation. It has been shown [12] that for parametric amplifiers $G = 2$ sets a borderline between quantum and classical performance of the device. For $G \geq 2$ an initially squeezed input loses the squeezing, i.e., its nonclassical properties in the process of amplification. This is sometimes called the ‘‘magic cloning limit’’; if it did not exist, one could reproduce quantum states, which would simply violate quantum mechanics. Meaning of the regimes below and above G_c can be discussed by considering an input where signals at each input port have the same amplitude but different phases at every second, e.g., $[1, 1, \omega^n, \omega^n, \omega^{2n}, \omega^{2n} \dots]/\sqrt{2N}$. If the input signals carry nonclassical features such as squeezing, then *both* inputs of each component B are fed with squeezed light; therefore the output of B can remain squeezed for

both $G_B < 2$ and $G_B > 2$ (for the cloning limit $G < 2$ one input is assumed to be the vacuum). In the next step the transforms A mix amplitudes with a *phase difference*; this means that a squeezed quadrature may become mixed with a nonsqueezed one. In this case the output can stay nonclassical only for $G_A < 2$. Thus G_c separates regions where the quantum-classical character of the network is or is not sensitive to the phases of the input. The sensitivity to the input phases can also be investigated by considering how K_n depends on ω^n below and above G_c . Note that the choice of the type of the input influences the interpretation since it has to be understood as a part of the system configuration. Here it should be also reminded that the transition can be understood as a quantum effect and not due to a finite temperature. Namely, the quantum theory of amplification implies loss of quantum information, that is, the SU(1,1) formulation utilized here, even for a zero-temperature reservoir because of the presence of quantum vacuum fluctuations in the amplification process [8,12].

Let us now consider the passive SU(2) networks [13]. The analogy to the Ising model is then not one to one. For example, (1) is true only for one trivial choice for the, now imaginary, angle ϕ . We can, however, define a network which has the same basic properties as the active ones: nodes connected by nearest neighbor interactions, with one parameter ϕ quantifying the importance of these interactions. Let us, for example, fix $\theta = i\pi/4$ and denote $\phi' = -i\phi$. In the limits $\phi' \rightarrow 0$ and $\phi' \rightarrow \pi/2$ the network decomposes into sets of independent modes, while intermediate values of ϕ' describe a network of interacting modes. The zeroth eigenvalue of the matrix P is defined by the equation $\cos \gamma_0 = \frac{1}{\sqrt{2}}(\cos \phi' + \sin \phi')$. It reaches the value zero for $\phi'_c = \pi/4$, and its derivative with respect to ϕ' has a singularity at this point. The transition thus takes place at the point when all the beam splitters are half transmitting. Names for the two regimes separated by ϕ'_c can be given by considering a network of avoided crossings. Imagine that in Fig. 1 the individual elements are actually avoided crossings between intersecting energy levels. At each crossing the system can either follow the energy level adiabatically, or make a Landau-Zener transition to the neighboring level, that is, to show diabatic behavior. Indicating which of these processes is more likely, we call the regime $\phi' < \pi/4$ diabatic and $\phi' > \pi/4$ adiabatic. Thus we have shown that the system does not evolve smoothly from the adiabatic to the diabatic regime and vice versa but exhibits at the critical transmittance $t_c = \frac{1}{2}$ a transition which is associated with singularities in the global parameters of the network.

We have now identified the regimes of behavior separated by the transition, both for the SU(1,1) and SU(2) networks. The essential feature is the appearance of singularities at the transition point. In a normal situation the tuning of local parameters, that is, the parameters of the network nodes such as beam splitters or amplifiers, leads to a smooth change in the global properties of the network.

At the transition point this is not true. To demonstrate the observability of this phenomenon we now consider the response of the network in the case of two generic types of input states: the equal superposition of all modes, and the eigenstate of one mode.

The output of the network is given by the transformation P^M . The M th powers of K_n in Eq. (2) can be obtained using the diagonalized form of K_n : $(K_n^M)_{11} = \cosh M\gamma_n + i \sin(2\pi n/N) \sinh M\gamma_n / \sinh \gamma_n = (K_n^M)_{22}^*$ and $(K_n^M)_{21} = (K_n)_{21} \sinh M\gamma_n / \sinh \gamma_n = (K_n^M)_{12}^*$. An input in an equal superposition state with a phase periodicity determined by n , that is, $[1, 0, \omega^n, 0, \omega^{2n}, \dots] / \sqrt{N}$, will be transformed into the output state $[(K_n^M)_{11}, (K_n^M)_{21}, \omega^n (K_n^M)_{11}, \omega^n (K_n^M)_{21}, \dots] / \sqrt{N}$. By choosing n one can thus control constructive and destructive interference in the network ($\gamma_{n+1} > \gamma_n$). The choice $n = 0$ is a special one: at G_c $\gamma_0 = 0$ and the whole network becomes transparent. The transparency remains true independent of M ; one may consider this to be analogous to the appearance of long range correlations at T_c in the Ising model. Furthermore, for $n = 0$ the transition is clearly manifested in the measurable output intensities. The global amplification coefficient is then simply $M\gamma_0$, and its rate of change with respect to the local amplification coefficient ϕ , $M \frac{d\gamma_0}{d\phi}$, has a discontinuity at G_c . For an input in the eigenstate of the $(2n + 1)$ th mode, i.e., $[0, 0, \dots, 1, \dots, 0]$, the $(2n + 1)$ th output amplitude has the form $\frac{1}{N} \sum_{m=0}^{N-1} \cosh M\gamma_m + i \sin(2\pi m/N) \sinh M\gamma_m / \sinh \gamma_m$. Since this sum contains all γ_m , the singularity is smoothed. By choosing the type of the input one can thus modify the manifestations of the transition in the output. Similar considerations can be carried out to find the fingerprints of the transition in the case of SU(2) networks.

Note that due to the nonsmoothness of γ_0 operations defined by G_c or t_c may be unstable ones. This has interesting consequences: for instance, some operations to be performed by the proposed quantum computer may hit an unstable transition point of the whole system; this is something to be aware of when designing complicated networks. Also quantum correlations and entanglement might show special behavior near the transition. Furthermore, it could be illuminating to consider higher dimensional quantum interference experiments in connection with the Ising model. The theoretical description of networks of avoided crossings [4,5,14] resembles the one presented here, with the choice $A = B$ and with the addition of extra phase factors due to free evolution of the system between the avoided crossings. However, when the free evolution phase factors are multiples of π/N , the system reduces to the one described here, up to trivial differences. It is interesting to note that experimentally observed *recurrence phenomena* appear in these networks exactly for such values of the free evolution phase, that is, when the system reduces to an Ising-type network. Moreover, approximate analytical results predicting the recurrences are possible to derive in

the adiabatic and diabatic limits [5], but not for the case of 50:50 crossings which according to our results is the transition point between these regimes.

In summary, we have taken a novel point of view to the networks used in quantum computation, communication, and interference experiments: we have shown that they can exhibit transition phenomena. Although the individual quantum components at the network nodes are smoothly behaving, in certain network configurations—like the 2D Ising model analog considered here—they show nonsmooth global behavior. For networks based on active SU(1,1) components the transition is particularly interesting since the critical amplification $G_c = 2$ coincides with the “magic cloning limit.” We have shown that for certain inputs the network has two regimes of quantum–classical behavior: below the transition point $G_c = 2$ the network is sensitive to the phase information in the input, above not. In the case of passive SU(2) networks the critical transmittance $t_c = \frac{1}{2}$ separates regions of adiabatic and diabatic behavior. One important point strikes out from the discussion of observable quantities: the manifestations of the transition are at their strongest when the input is totally symmetric (e.g., $[1, 0, 1, 0, \dots] / \sqrt{N}$). In this case the combined network–input configuration is totally symmetric at G_c , and below and above G_c the roles of the transforms A and B are interchanged. Discontinuities at G_c are therefore associated with breaking of a symmetry. Thus the analysis presented in this Letter offers a working hypothesis for the study of more general quantum networks: a quantum network might show transition phenomena whenever some values of its functioning and input parameters lead to creation of a certain degree of symmetry in the system.

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