Integrability of Three-Particle Evolution Equations in QCD

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We show that Brodsky-Lepage evolution equation for the spin-3/2 baryon distribution amplitude is completely integrable and reduces to the three-particle $XXX_{s=-1}$ Heisenberg spin chain. Trajectories of the anomalous dimensions are identified and calculated using the $1/N$ expansion. Extending this result, we prove integrability of the evolution equations for twist-3 quark-gluon operators in the large N_c limit. [S0031-9007(98)07031-8]

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Three-particle distribution amplitudes (DAs) appear in QCD in various contexts and attract increasing attention. Perhaps the most important example is provided by the baryon DA which specifies momentum fraction distribution between the three valence quarks in the minimal Fock state and is defined by the baryon-to-vacuum transition matrix element of the nonlocal three-quark operator [1]:

$$B(a,b,c) = \varepsilon_{ijk} q^i(au) q^j(bu) q^k(cu) . \qquad (1)$$

Here q is a quark field and u is an auxiliary lightlike vector $u^2 = 0$; taking the leading-twist part and insertion of gauge factors is implied. Another example is provided by twist-3 quark-gluon operators

$$S_\mu^\pm(a,b,c) = \bar{q}(au)[iG_{\mu\nu}(bu) \pm \tilde{G}_{\mu\nu}(bu)\gamma_5]u^\nu \not{u} q(cu) , \qquad (2)$$

$$T(a,b,c) = \bar{q}(au) u^\mu u^\nu \sigma_\mu^\rho G_{\nu\rho}(bu) \Gamma q(cu) , \qquad (3)$$

where $\Gamma = \{I, i\gamma_5\}$, which give rise to twist three nucleon parton distributions and/or twist-3 DAs of vector mesons.

Parton distributions in QCD are scale dependent; for their *moments* this dependence corresponds to renormalization group (RG) behavior of the relevant local operators obtained by expansion at short distances. This connection is well understood and worked out in great detail for two-particle operators corresponding to leading twist nucleon parton densities or meson DAs. In this case a single independent local operator exists for each moment. The corresponding anomalous dimension can be continued analytically to noninteger (complex) moments, defining the Altarelli-Parisi evolution kernel. Mixing with total derivatives can be resolved to one-loop accuracy by going over to the conformal basis [2–4]: coefficients in the expansion of meson DAs in Gegenbauer polynomials are renormalized multiplicatively and with the same anomalous dimensions as in deep inelastic scattering.

Three-particle distributions bring in a complication of principle. For fixed operator dimension alias fixed total number N of covariant derivatives $N + 1$ independent local operators exist corresponding to geniune degrees of freedom. One is left with a nontrivial $(N + 1) \times (N + 1)$ mixing matrix and has to diagonalize it explicitly order by order; see, e.g., [4–6]. The resulting $N + 1$ multiplicatively renormalizable operators have different (in general) anomalous dimensions which analytic expressions are not known (see, however, [7,8]). Apart from mathematical incompleteness, absence of analytic results means that the general structure of the spectrum is unknown and, in particular, analytic continuation of the anomalous dimensions to noninteger N is not possible. This, in turn, implies that partonic interpretation of different "components" is not fully understood beyond the tree level.

Main result of this Letter is that the three-particle Schrödinger equation describing the renormalization of spin 3/2 baryon operators (1) is completely integrable, i.e., has a nontrivial integral of motion. In addition, we prove integrability of the RG equations for twist-3 quark-gluon operators (2)–(3) in the limit of a large number of colors N_c. Our finding is similar in spirit to the recent discovery [9,10] of integrability of the system of interacting reggeized gluons in QCD, but is obtained in a different context. A physical interpretation is that we identify a new "hidden" quantum number which distinguishes components in three-particle parton distributions with different scale dependence. It turns out that the evolution equation for spin 3/2 baryon operators is related to the equation for the odderon trajectory, and the results obtained in the latter context [11,12] can be adapted to unravel the spectrum of baryon operators. Using this connection, we identify trajectories of the anomalous dimensions and calculate them using the $1/N$ expansion. Explicit formulas are given for the highest and lowest anomalous dimensions

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in the spectrum. Further applications will be presented elsewhere [13].

To one-loop accuracy, the divergent part of the nonlocal operator $\Phi(a_1,a_2,a_3)$ where $\Phi = B, S^{\pm}, T$, has the form $(1/\epsilon)H\Phi(a_i)$, and the explicit expression for the integral operator H is known for all cases under consideration [6,8,14]. An arbitrary local operator $\mathcal{O}$ with N covariant derivatives can be represented by the associated polynomial in three variables $\psi(a_1,a_2,a_3)$ of degree N such that $\mathcal{O}_\psi = \psi(\partial_a,\partial_b,\partial_c)\Phi(a,b,c)_{a,b,c\to 0}$ where $\partial_a = \partial/\partial a$, etc. In order to find multiplicatively renormalizable local operators one has to solve the Schrödinger equation for the ψ functions, $\widetilde{H}\psi = \mathcal{E}\psi$, where $\widetilde{H}$ is easy to find if H is given.

It proves convenient to define the integral transformation [15] $\psi(a_i)\to\hat{\psi}(z_i)$ by

$$\hat{\psi}(z_i) \equiv \prod_i \int_0^\infty dt_i\, \frac{e^{-t_i} t_i^{l_i+s_i-1}}{\Gamma(l_i+s_i)}\, \psi(z_1t_1,\ldots,z_3t_3)\,, \tag{4}$$

where l_i and s_i are the canonical dimension and spin projection of the ith field, respectively: $l=3/2$, $s=1/2$ for quarks (antiquarks) and $l=2$, $s=1$ for gluons. We can reformulate the above eigenvalue problem in terms of $\hat{\psi}$ functions, and it is easy to check that the corresponding Hamiltonian $\hat{H}$ coincides with the initial Hamiltonian for the nonlocal operator, $\hat{\widetilde{H}} \equiv H$. Note that H has a two-particle structure, $H=\sum_{i,k}H_{ik}$. Conformal invariance implies that the two-particle Hamiltonians H_{ik} commute with SL(2) generators $J^{\pm,3} = \sum_{i=1}^3 J_i^{\pm,3}$, where

$$J_i^+ = z_i^2\partial_i + (l_i+s_i)z_i\,, \qquad J_i^- = -\partial_i\,, \qquad J_i^3 = z_i\partial_i + (l_i+s_i)/2\,, \tag{5}$$

and are Hermitean with respect to the scalar product

$$\langle\hat{\psi}_1|\hat{\psi}_2\rangle = \psi_1(\partial_1,\ldots,\partial_3)\hat{\psi}_2(z_1,\ldots,z_3)|_{z_i=0}\,. \tag{6}$$

Thus, the equation $H\hat{\psi} = \mathcal{E}\hat{\psi}$ decays into the set of eigenvalue problems on the subspaces of functions with fixed value of J_3, $J_3\hat{\psi} = j_3\hat{\psi}$ and annihilated by J_-, $J_-\hat{\psi} = 0$. The latter condition is simply shift invariance [15]. Therefore, eigenfunctions of two-particle Hamiltonians are given by simple powers $\hat{\psi}_l = (z_i - z_k)^l$ instead of Jacobi polynomials in standard variables [3].

The SL(2) invariance imposes stringent restrictions on the form of two-particle operators, so that only a few structures are allowed. One such structure corresponds to the "vertex correction" involving the gluon field from (one of) the covariant derivatives (in Feynman gauge):

$$H_{12}^v\hat{\psi}(\underline{z}) = -\int_0^1 \frac{d\alpha}{\alpha}\,\{\overline{\alpha}^{l_1+s_1-1}[\hat{\psi}(z_{12}^\alpha,z_2,z_3) - \hat{\psi}(\underline{z})] + \overline{\alpha}^{l_2+s_2-1}[\hat{\psi}(z_1,z_{21}^\alpha,z_3) - \hat{\psi}(\underline{z})]\}\,, \tag{7}$$

where $\underline{z}\equiv\{z_1,z_2,z_3\}$, $z_{ik}^\alpha = z_i\overline{\alpha} + z_k\alpha$, and $\overline{\alpha} = 1-\alpha$. Another structure originates from gluon exchange between quarks (or between a quark and a gluon):

$$H_{12}^e\hat{\psi}(\underline{z}) = 2\int_0^1 D\underline{\alpha}\, \frac{\overline{\alpha}_1^{l_1+s_1}\overline{\alpha}_2^{l_2+s_2}}{(\overline{\alpha}_1\overline{\alpha}_2)^2}\, \hat{\psi}(z_{12}^{\alpha_1}, z_{21}^{\alpha_2}, z_3)\,, \tag{8}$$

where $D\underline{\alpha} \equiv \prod_{i=1}^3 d\alpha_i\,\delta(1-\sum\alpha_i)$.

Because of SL(2) invariance the two-particle Hamiltonians must depend on the corresponding Casimir operators $L_{ik} \equiv (\vec{J}_i + \vec{J}_k)^2 = J_{ik}(J_{ik}-1)$ only. This dependence can be easily reconstructed from the spectrum of H_{ik}. Since the form of the eigenfunctions is known $\hat{\psi}_l = (z_i - z_k)^l$, it is straightforward to derive

$$\begin{aligned} H_{ik}^{v,(qq)} &= 2[\psi(J_{ik}) - \psi(2)]\,, \\ H_{ik}^{v,(qg)} &= \psi(J_{ik}+1/2) + \psi(J_{ik}-1/2) \\ &\quad - \psi(3) - \psi(2)\,, \end{aligned} \tag{9}$$

where $\psi(x)\equiv\Gamma'(x)/\Gamma(x)$. The superscripts (qq) and (qg) indicate quark-quark and quark-gluon operators, respectively. Similarly, we obtain

$$\begin{aligned} H_{ik}^{e,(qq)} &= 2J_{ik}^{-1}(J_{ik}-1)^{-1}\,, \\ H_{ik}^{e,(qg)} &= 2(J_{ik}-3/2)^{-1}(J_{ik}+1/2)^{-1}\,. \end{aligned} \tag{10}$$

We are now in a position to specify RG equations for the operators in (1)–(3) explicitly. One has to distinguish three-quark operators belonging to $(3/2,0)$ and $(1,1/2)$ representations, which correspond to DAs for spin 3/2 and spin 1/2 baryons, respectively. We get [16]:

$$H_{3/2} = H_{12}^{v,(qq)} + H_{13}^{v,(qq)} + H_{23}^{v,(qq)}\,, \tag{11a}$$

$$H_{1/2} = H_{3/2} - (1/2)H_{12}^{e(qq)} - (1/2)H_{23}^{e,(qq)}\,. \tag{11b}$$

Omitting subleading in N_c terms, the quark-antiquark-gluon Hamiltonians are [7,8]

$$H_{S^+} = H_{12}^{v,(qg)} + H_{23}^{v,(gq)} - H_{12}^{e,(qg)}\,, \tag{12a}$$

$$H_{S^-} = H_{12}^{v,(qg)} + H_{23}^{v,(gq)} - H_{23}^{e,(gq)}\,, \tag{12b}$$

$$H_T = H_{12}^{v,(qg)} + H_{23}^{v,(gq)} - H_{12}^{e,(qg)} - H_{23}^{e,(gq)}\,. \tag{12c}$$

The properly defined anomalous dimensions are given in terms of eigenvalues of the above operators including color factors and trivial contributions of self-energy insertions:

$$\begin{aligned} \gamma_{3/2,1/2}(N) &= (1+1/N_c)\mathcal{E}_{3/2,1/2}(N) + (3/2)C_F\,, \\ \gamma_{S,T}(N) &= N_c\mathcal{E}_{S,T}(N) + (7/2)N_c \end{aligned} \tag{13}$$

where $C_F = (N_c^2-1)/(2N_c)$.

The operators $H_{S^\pm}$ are equivalent; hereafter, we consider H_{S^+}. The $1/N_c^2$ corrections to (12) and RG equations for three-gluon operators involve additional structures [13] and will not be discussed here.

We have been able to find integrals of motion Q_i, $[H_i,Q_i]=0$, for all Hamiltonians in question with the exception of $H_{1/2}$. Explicit expressions for the conserved charges Q_i present the main result of this Letter:

$$Q_{3/2} = \imath[L_{12}, L_{13}] = \imath\partial_1\partial_2\partial_3 z_{12}z_{23}z_{31}\,, \tag{14}$$

$$Q_{S^+} = \{L_{12}, L_{23}\} - \frac{9}{2}L_{23} - \frac{1}{2}L_{12}\,, \tag{15}$$

$$Q_T = \{L_{12}, L_{23}\} - \frac{9}{2}L_{12} - \frac{9}{2}L_{23}\,, \tag{16}$$

where $\{,\}$ stands for an anticommutator. Remarkably, $H_{3/2}$ is nothing but the familiar Hamiltonian of the $XXX_{s=-1}$ three-particle Heisenberg spin chain. The expression in (14) for the corresponding conserved charge $Q_{3/2}$ is well known; see, e.g., [10].

To check that $[H_T, Q_T] = 0$, $[H_S, Q_S] = 0$ it is convenient to introduce the basis of functions [15] $\hat{\psi}_{12}^{N,j}(z_1, z_2; z_3)$ which diagonalize the full three-particle Casimir operator $L_{123} = (\vec{J}_1 + \vec{J}_2 + \vec{J}_3)^2$ and the two-particle Casimir in the (1,2)-channel simultaneously:

$$\begin{aligned} L_{123}\hat{\psi}_{12}^{N,j} &= (N + 7/2)(N + 5/2)\hat{\psi}_{12}^{N,j}, \\ L_{12}\hat{\psi}_{12}^{N,j} &= j(j - 1)\hat{\psi}_{12}^{N,j}. \end{aligned} \tag{17}$$

A remarkable property of this basis is that the other two two-particle Casimir operators become three-diagonal, i.e., $\langle j|L_{23}|j'\rangle \neq 0$ for $|j - j'| \leq 1$ only. Explicit expressions for $\hat{\psi}_{12}^{N,j}$ can easily be constructed in terms of hypergeometric functions [13]. Using them, it is easy to derive that

$$[Q_T, H_T^{(12)}] = 2[L_{12}, L_{23}]\,, \tag{18}$$

where $H_T^{(12)} = H_{12}^{v,(qg)} - H_{12}^{e,(qg)}$ and, similarly, $[Q_T, H_T^{(23)}] = 2[L_{23}, L_{12}]$. Hence $[Q_T, H_T] = 0$. The proof of the relation $[H_S, Q_S] = 0$ is analogous.

Once conserved charges are known, one can consider the eigenvalue problem for these charges instead of the Hamiltonians, which is simpler. For the Heisenberg spin chain, a detailed study exists due to Korchemsky [11,12]. The spectrum of $Q_{3/2}$ is shown in Fig. 1a. For generic integer N there exist $N + 1$ eigenvalues which come in pairs $\pm q$. Note that for even N $Q_{3/2}$ has zero eigenvalue $q = 0$. The corresponding value of energy can be calculated exactly:

$$\mathcal{E}_{3/2}(N, q = 0) = 4\psi(N + 3) + 4\gamma_E - 6\,; \tag{19}$$

see the dotted curve in Fig. 1b. Eigenvalues of $Q_{3/2}$ lie on trajectories which were found in [11] using a "semiclassical" expansion in the conformal spin $h = N + 3$:

$$\begin{gathered} q(N,k)/h^3 = \sum_m q^{(m)}(k)/h^m, \\ q^{(0)} = 1/\sqrt{27}\,, \qquad q^{(1)}(k) = -(k + 1)/\sqrt{3}\,, \ldots\,. \end{gathered} \tag{20}$$

The $q^{(m)}(k)$ are polynomials of degree m; the first eight of them are given in Eq. (5.14) in Ref. [11]. k is a nonnegative integer which numerates the trajectory. The asymptotic expansion in (20) is valid for $q > 0$ only and the analytic continuation of the trajectory to $q < 0$ can be obtained by using symmetry properties of the solutions [11]: $q(N,k) \to -q(N, N - k)$.

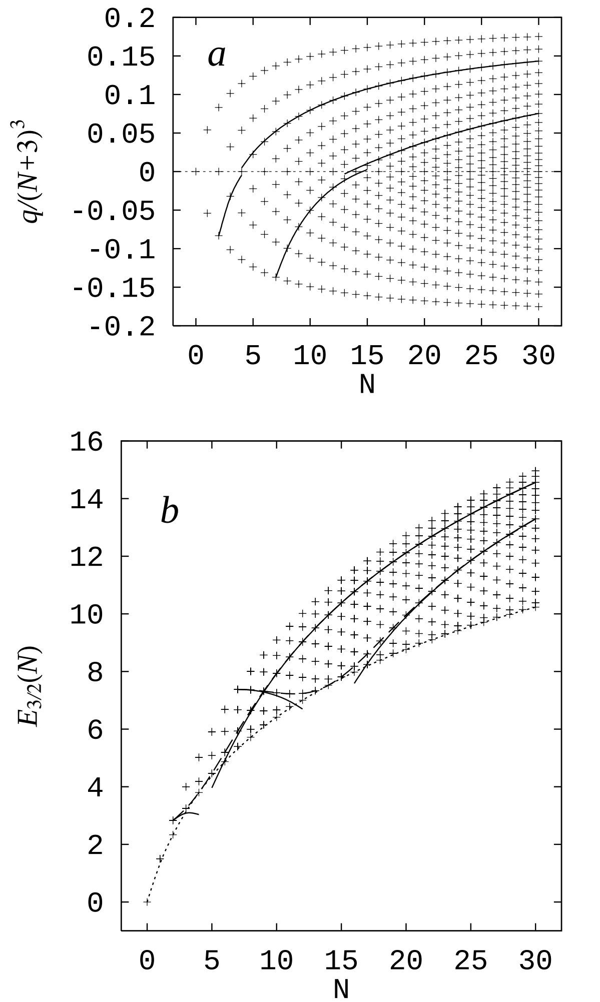

FIG. 1. The spectrum of eigenvalues for the conserved charge $Q_{3/2}$ (a) and for the spin 3/2 Hamiltonian $H_{3/2}$ (b), see text.

Two examples of the trajectories with $k = 2$ and $k = 7$ are shown in Fig. 1a together with exact eigenvalues (crosses) calculated numerically. Note that the two asymptotic expansions—for positive and negative q—match reasonably well. Analytic expressions for the trajectories in the $q \to 0$ region are available from [12].

The low-lying eigenvalues of $q(N,k)$ can be calculated to $O(1/N^2)$ accuracy from the equation [13]

$$\bar{q}\ln(h - 1/2) - \arg[\Gamma(1 + i\bar{q})] = \frac{\pi}{6}(N - 2k)\,, \tag{21}$$

where $\bar{q} \equiv q/h(h - 1)$, which is valid for $k - N/2 \ll N$. The lowest value of $|q|$ for odd N is thus of order

$$\bar{q} = \pm\frac{\pi}{6}[\ln(h - 1/2) + \gamma_E]^{-1} + O(1/\ln^4 h)\,. \tag{22}$$

The spectrum of $H_{3/2}$ is shown in Fig. 1b. Exact eigenvalues obtained by explicit diagonalization of the mixing matrix are shown by crosses. Since $\mathcal{E}_{3/2}(q) = \mathcal{E}_{3/2}(-q)$ all eigenvalues except for the ones for $q = 0$ are double degenerate. The energy eigenvalues lie on trajectories corresponding to the trajectories for q in Fig. 1a, and, similar to the latter, can be calculated using a "semiclassical" expansion [11]

$$\mathcal{E}_{3/2}(N,k) = \varepsilon^{(0)} - \sum_{m=1}^{\infty} \varepsilon^{(m)}(k)/h^m, \qquad \varepsilon^{(0)} = 6\ln(N+3) + 6\gamma_E - 6 - 3\ln 3, \ldots . \tag{23}$$

The polynomials $\varepsilon^{(m)}(k)$ are given in Eq. (6.5) of Ref. [11] up to $m = 7$. The trajectories corresponding to $k = 2$ and $k = 7$ are shown in Fig. 1b by broken lines, whereas the solid curves correspond to the asymptotic expansion in (23) [17]. Note that the expansion diverges close to the "deflection points" which occur at even integer N and with the energy given by Eq. (19). Convergence of the $1/h$ expansion is somewhat worse for the energy compared to the conserved charge q, but it can be improved systematically. Alternatively, one can derive analytic approximations for $q(N,k)$ and $\mathcal{E}_{3/2}(q)$ applicable in the $q \to 0$ region; see [12,13].

Using (22) we obtain an estimate for the lowest energy eigenvalue for odd N:

$$\mathcal{E}_{3/2}(N) = 4\ln N - 6 + 4\gamma_E + \frac{\zeta(3)}{18\ln^2 N}. \tag{24}$$

Numerically, the difference to Eq. (19) is very small, compare exact eigenvalues with the dotted curve in Fig. 1b, and is probably irrelevant for phenomenological applications. One has to bear in mind, however, that an approximation of taking into account operators with the lowest anomalous dimension only for each N is theoretically inconsistent since they belong to different trajectories.

The anomalous dimensions of quark-gluon operators (15), (16) can be studied along similar lines [13]. To the $O(1/N)$ accuracy the spectrum of low-lying energy eigenvalues is given in terms of eigenvalues of the corresponding integrals of motion as

$$\mathcal{E}(\nu) = 2\ln N + 4\gamma_E - 5 + 2\,\mathrm{Re}[\psi(3/2 + i\nu)], \tag{25}$$

where $2\nu_{S,T}^2 = q_{S,T} - 3/2$, and quantization conditions for the effective charges read, to the same accuracy

$$\nu_T \ln N + \Phi_1(\nu) - \Phi_3(\nu) = \frac{\pi n}{2}, \qquad \nu_S \ln N + \frac{1}{2}[\Phi_1(\nu) + \Phi_2(\nu)] - \Phi_3(\nu) = \frac{\pi n}{2}, \tag{26}$$

where $n = 1, 2, \ldots$ and

$$\Phi_1(\nu) = (1/2)\arg[{}_2F_1^2(3/2 + i\nu, -3/2 + i\nu, 1 + 2i\nu, 1)], \quad \Phi_2(\nu) = \arg[{}_2F_1(1/2 + i\nu, -1/2 + i\nu, 1 + 2i\nu, 1)], \quad \Phi_3(\nu) = \arg[\Gamma(3/2 + i\nu)]. \tag{27}$$

These formulas are not applicable to the exact solutions with minimum anomalous dimensions, found in Refs. [7,8], which correspond to imaginary ν and have to be treated separately. These special solutions are separated from the rest of the spectrum by a finite gap. A detailed study is in progress.

To summarize, we have shown that a few important three-particle evolution equations in QCD are exactly integrable; that is, they possess nontrivial integrals of motion. This gives a fairly complete description of the spectrum of anomalous dimension of baryon operators with spin $3/2$, and a similar description can be developed for the quark-gluon operators as well. The eigenfunctions can also be studied [12,13]. We believe that the approach based on integrability may find many phenomenological applications to studies of higher-twist parton distributions in QCD.

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[17] The intersecting trajectories pick up different eigenstates from the degenerate pairs.