Jamming, Force Chains, and Fragile Matter

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We consider materials whose mechanical integrity is the result of a jamming process. We argue that such media are generically "fragile," unable to support certain types of incremental loading without plastic rearrangement. Fragility is linked to the marginal stability of force chain networks within the material. It can lead to novel mechanical responses that may be relevant to (a) jammed colloids and (b) poured sand. The crossover from fragile to elastoplastic behavior is explored. [S0031-9007(98)06815-X]

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Consider a concentrated colloidal suspension of hard particles under shear [Fig. 1(a)]. Above a certain threshold of stress, this system may jam [1]. (To observe such an effect, stir a concentrated suspension of cornstarch with a spoon.) Jamming apparently occurs because the particles form "force chains" along the compressional direction [1]. Even for spherical particles the lubrication films cannot prevent contacts; once these arise, an array or network of force chains can support the shear stress indefinitely [2]. By this criterion, the material is a solid. In this Letter, we propose some simple models of jammed systems like this, whose solidity stems directly from the applied stress itself. We argue that such materials may show fundamentally new mechanical properties, very different from those of conventional (elastic or elastoplastic) bodies.

We start from a simple model of a force chain: a linear string of rigid particles in point contact. Crucially, this chain can only support loads *along its own axis* [Fig. 2(a)]: successive contacts must be collinear, with the forces along the line of contacts, to prevent torques on particles within the chain [3]. (Neither friction at the contacts nor particle asphericity can obviate this.)

Let us now model a jammed colloid by an assembly of such force chains, characterized by a director $\mathbf{n}$, in a sea of "spectator" particles, and incompressible solvent. (We ignore for the moment any "collisions" between force chains or deflections caused by weak interaction with the spectators.) In static equilibrium, with no body forces acting, the pressure tensor $p_{ij}(= -\sigma_{ij})$ is then

$$p_{ij} = P\delta_{ij} + \Lambda n_i n_j\,, \tag{1}$$

where P is an isotropic fluid pressure, and Λ (>0) a compressive stress carried by the force chains.

Even this minimal model of the jammed state exhibits quite novel mechanical properties. Indeed, Eq. (1) permits static equilibrium only so long as the applied compression is along $\mathbf{n}$; while this remains true, small, or even large, incremental loads can be accommodated reversibly, by what is (ultimately) an elastic mechanism. But the material is certainly not an elastic body, for if instead one tries to shear the sample in a slightly different direction (causing a rotation of the principal stress axes) static equilibrium cannot be maintained without changing the director $\mathbf{n}$. And since $\mathbf{n}$ describes force chains that pick their ways through a dense sea of spectator particles, it cannot simply rotate; instead, the existing force chains must be abandoned and new ones created with a slightly different orientation. This entails dissipative, plastic, reorganization, during which the system will rejam to support the new load. (The system resembles a liquid crystal, except that the stress causes transient rearrangement, not steady flow.)

The jammed colloid is an example of *fragile matter*: it can statically support applied shear stresses (within some range), but only by virtue of a self-organized internal structure, whose mechanical properties have evolved to support the load itself. Its incremental response can be elastic only to *compatible* loads; *incompatible* loads (in this case, those of a different compression axis), even if small, will cause finite, plastic reorganizations. The inability to elastically support *some* infinitesimal loads is our definition of "fragile" (and more precise than any we have previously seen). It extends naturally to other perturbations; e.g., small changes in temperature which can lead to "static avalanches" of rearrangement [4].

We now argue that jamming may lead *generically* to fragile matter (as defined above). If a system arrests as soon as it can support the external load, its state is

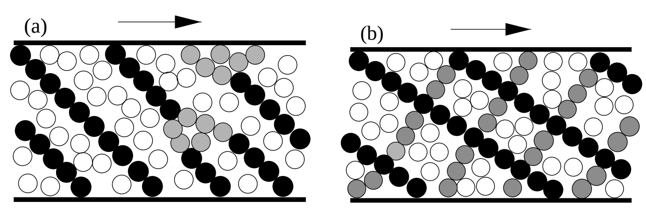

FIG. 1. (a) A jammed colloid (schematic). Black: force chains; grey: other force-bearing particles; white: spectators. (b) Idealized rectangular network of force chains.

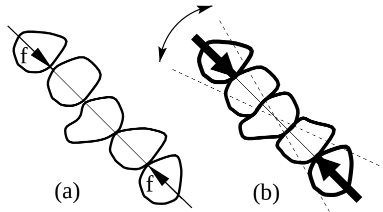

FIG. 2. (a) A force chain of hard particles (any shape) can statically support only longitudinal compression. (Body forces acting directly on these particles are neglected.) (b) Finite deformability allows small transverse loads to arise.

likely to be only marginally stable. Incompatible perturbations force rearrangement, leaving the system in a newly jammed, equally fragile, state. This scenario is related to suggestions that rigidity emerges by successive buckling of force chains (impeded by spectators) in glasses and granular matter [5]. It also resembles self-organized criticality (SOC) [6]; we return to this later (see also [4,7]). However, we focus first on simple models of the fragile state in static equilibrium.

Consider again our (homogeneously) jammed colloid. What body forces can it now support *without* plastic rotation of the director? Various models are possible. One is to assume that Eq. (1) continues to apply, with $P(\mathbf{r})$ and $\Lambda(\mathbf{r})$ now varying in space. If P is a simple fluid pressure, a localized body force can be supported only if it acts along $\mathbf{n}$. Thus (as in a bulk fluid) no static Green function exists for a general body force. To support the latter in three dimensions, in fact, requires more than one orientation of force chain, perhaps forming a network or skeleton [8–11]. A simple model for this is

$$p_{ij} = \Lambda_1 n_i n_j + \Lambda_2 m_i m_j + \Lambda_3 l_i l_j \,, \qquad (2)$$

with $\mathbf{n}, \mathbf{m}, \mathbf{l}$ directors along three nonparallel populations of force chains; the Λ's are compressive pressures acting along these. Body forces cause $\Lambda_{1,2,3}$ to vary in space.

We can thus distinguish different levels of fragility, according to whether incompatible loads include localized body forces [*bulk* fragility, e.g., Eq. (1)], or are limited to forces acting at the boundary [*boundary* fragility, e.g., Eq. (2)]. In disordered systems one might also distinguish between macrofragile responses involving changes in the *mean* orientation of force chains, and the microfragile responses of individual contacts. Below we focus on macrofragility, but if the medium shows static avalanches [4] the distinction may become blurred.

Returning to the simple model of Eq. (2), the chosen values of the three directors (two in 2D) clearly should depend on how the system came to be jammed (its "construction history"). If it jammed in response to a constant external stress, switched on suddenly at some earlier time, one can argue that the history is specified *purely by the stress tensor itself*. In this case, if one director points along the major compression axis, then by symmetry any others should lie at right angles to it [Fig. 1(b)]. Applying a similar argument to the intermediate axis leads to the ansatz that all three directors lie along principal stress axes; this is perhaps the simplest model in 3D [12]. (One version of this argument links force chains with the fabric tensor [10], which is then taken coaxial with the stress [11].) If so, Eq. (2) does not change form if an arbitrary isotropic pressure field P is added. One recovers Eq. (1) in 2D, and in cases of uniaxial symmetry.

With the ansatz of perpendicular directors as just described, Eq. (2) becomes a "fixed principle axes" (FPA) model [7,13]. Although grossly oversimplified, this leads to nontrivial predictions for the jammed state. For example, in an idealized colloidal jamming experiment [1,14], one imposes a fixed spacing between parallel plates lying in the (x, z) plane, and a fixed shear stress σ_{xy} at the plates. The normal stress differences $N_1 = \sigma_{xx} - \sigma_{yy}$ and $N_2 = \sigma_{yy} - \sigma_{zz}$ are then monitored (also the shear rate $\dot{\gamma}$, which should vanish in a jammed state). The FPA model then contains, as a parameter, the angle φ between the major compression axis $\mathbf{n}$ and the y direction. The model gives $\sigma_{xy} = [\Lambda_1 - \Lambda_2]sc$, $N_1 = [\Lambda_2 - \Lambda_1](s^2 - c^2)$, and $N_2 = \Lambda_3 - \Lambda_1 c^2 - \Lambda_2 s^2$ where $s, c = \sin\varphi, \cos\varphi$. (The 3 axis is along z.) So, FPA predicts a constant ratio $\alpha \equiv -N_1/\sigma_{xy} = (s^2 - c^2)/sc$ as σ_{xy} is varied within a given jammed state. Remarkably, recent experiments [14] do report such constancy of α (with $\alpha \simeq 1.0 \pm 0.1$, or $\varphi = 58°$) within "the regime of strong shear thickening" (in which σ_{xy} shows time dependence reminiscent of stick-slip behavior [14]). Although not strictly a jammed state ($\dot{\gamma} > 0$ on average) the success of the FPA concept in this regime is striking [15].

We now turn from colloids to granular materials. Although the formation of dry granular aggregates under gravity is not normally described in terms of jamming, it is closely related. Indeed, the filling of silos and the motion of a piston in a cylinder of grains both exhibit jamming and stick-slip phenomena associated with force chains; see [16]. Moreover, FPA-like models account quite well for the forces measured experimentally beneath conical piles of sand, constructed by pouring cohesionless grains from a point source onto a rough rigid support [7,13,17]. Hence fragile models of granular media must merit serious consideration. They share some features with recent *hypoplastic* models of such media [10].

Note that in 2D, when combined with stress continuity ($\partial_i \sigma_{ij} = \rho g_j$ for sand under gravity), Eq. (2) gives differential equations for the stress tensor which are hyperbolic [7,13,18]. With a zero-force boundary condition at the upper surface of a pile [7], this gives a well-posed problem: the forces acting at the base follow uniquely from the body forces by integration. [Analogous remarks apply to Eq. (2) in 3D.] If *different* forces are now imposed at the base, rearrangement is inevitable. (This is boundary-fragile behavior.) The same does not hold [19] within a traditional elastoplastic modeling approach [20] whose equations are elliptic in elastic zones and hyperbolic in plastic ones. In the sandpile, where an elastic zone contacts the support, the forces acting at the base cannot be found without specifying a displacement field there. To define this displacement, one would normally invoke as *reference state* the one in which the load (gravity) is removed. For cohesionless poured sand, this state is undefined [21], just as it is for a jammed colloid which, in the absence of the applied shear stress, is a fluid.

This "elastic indeterminacy" of sandpiles has no facile resolution [19,21]. One avenue is to consider a hypothetical sandpile where each grain becomes firmly "glued" to its neighbors (or the base), upon first coming to rest. The resulting medium is surely elastic, but contains quenched stresses arising from the addition of new material to the deformed pile as it is built [19]. (Put differently, were the grains to pack into a crystal, it would have a finite dislocation density.) In such a medium the displacement field is not single valued, and the solution of the elastic problem, though possible in principle, requires the whole construction history to be taken into account. Almost all elastoplastic calculations seem to ignore this. More importantly, for a typical disordered packing of near-rigid, glued grains, there will arise many *tensile contacts* even under a purely compressive external load [22]. Hence, even were realistic elastic (or elastoplastic) calculations available, their relevance to real (unglued) sandpiles, in which tensile contacts are entirely forbidden, remains in doubt [23]. The nonexistence of the standard (zero load) reference state arises *precisely when* ordinary mechanical behavior could give way to fragility: in systems whose solidity arises solely because of the applied load itself. This points towards our alternative, fragile description of cohesionless, poured sand under gravity.

Nonetheless, one can anticipate a crossover between fragile and elastic or elastoplastic behavior. Sand under strong enough compression may become elastic; even in unconsolidated, poured sand, sound waves of *sufficiently* small amplitude might propagate normally (although this is actually far from obvious experimentally [24]). Likewise in our jammed colloid, *extremely small* rotations of the principal axes might be accommodated elastically.

We next show, for a specific example of a fragile granular skeleton, that just such a crossover can arise from *slight particle deformability*. We consider a highly idealized, 2D rectangular skeleton of rigid particles, Fig. 1(b). In this material, where the longitudinal compressive forces balance at each node [25], the shear stress must vanish across planes parallel to **n** and **m** (that is, $p_{nm} = p_{mn} = 0$). For simplicity we also assume that the ratio Λ_1/Λ_2 (and its inverse) cannot exceed some constant K (for example, to avoid buckling of the stress paths). This implies a Coulomb inequality, $|p_{qr}| \leq p_{qq} \tan\phi$, for all other orthogonal unit vector pairs $\mathbf{q}, \mathbf{r}$; here $\tan\phi$ is a material constant such that $K = (1 - \sin\phi)/(1 + \sin\phi)$.

Now a small degree of particle deformability is introduced. This relaxes slightly the collinearity requirement of forces along chains, because the point contacts between particles are now flattened [Fig. 2(b)]. Clearly the ratio ϵ of the maximum transverse load to the normal one will vanish in some specified way (dependent on contact geometry) with the mean particle deformation. The same ratio ϵ defines, in effect, the maximum elastic angular deviation of the force chains. The system can thus be described as an (anisotropic) elastic body subject to a yield criterion of the following form:

$$|p_{qr}| \leq p_{qq} \tan\Phi(\mathbf{q} \cdot \mathbf{n}), \qquad (3)$$

where $\Phi(x)$ is a smooth function that is small (of order ϵ) in a narrow range (of order ϵ wide) of orientations around $x = 0$ (and $x = 1$), but close to ϕ outside this interval.

For finite ϵ this material will have mixed elliptic/hyperbolic equations of the usual elastoplastic type. But the resulting elastic and plastic zones must arrange themselves so as to obey the fragile model to within terms that vanish as $\epsilon \to 0$. If, in a sandpile, ϵ is small but finite, then stresses will depend on the detailed boundary conditions at the base of the pile, but only through small corrections to the leading, fragile result (FPA in this example). These deviations can accommodate an elastic response to very small incremental loads (on a scale set by ϵ). But for the macroscopic stress pattern to differ significantly from the hyperbolic prediction, one requires *appreciable particle deformation*. When the mean stresses are large enough to cause this ($\epsilon \simeq 1$), "ordinary" elastic or elastoplastic behavior will be recovered. Conversely, the fragile, hyperbolic limit emerges as *the limit of high particle rigidity* for this simplified model skeleton. Thus fragile models of granular or jammed matter, properly interpreted, need not contradict (though equally they do not require) an underlying elastoplastic description.

How valid are these new ideas? For granular media, the existence of tenuous force-chain skeletons is clear [8–11,26]. Simulations of frictional spheres show most of the deviatoric stress to arise from force chains; interparticle shear forces and "spectator" contacts provide mainly an isotropic pressure [9,11]. [Of course, the specific geometry of Fig. 1(b) is grossly oversimplified: the force chains are anisotropic, but not straight, with frequent collisions [9].] Several arguments suggest that such granular skeletons are close to the fragile limit. (Note that this limit does not involve a critical packing density, but a marginal integrity of the granular skeleton related to the absence of tensile forces [22].) First, the probability distribution for interparticle forces $p(f)$ does not vanish at zero force [11,26]. This is consistent with the idea that a *small* incompatible load (relative scale $\delta/\bar{f}$ with $\bar{f}$ the mean interparticle force) can induce a fraction $p(0)\delta$ of contacts to switch from spectator type ($f \simeq 0$) to force-chain type ($f \simeq \bar{f}$). The effect of this would be comparable with the elastic response ($\bar{f} \to \bar{f} \pm \delta$), and formally destroy the elastic regime. Second, simulations show strong rearrangement under small changes of compression axis; the skeleton is indeed "self-organized" [9,11]. Experiments also suggest cascades of rearrangement [4,16] in response to small disturbances. The latter is reminiscent of SOC [6], to which our suggestion of *generic* fragility in jammed states is closely related. SOC-like ideas also underlie recent discussions of dynamic attractors in hypoplastic models [10], and are not far removed from the (much older) critical state theories of soil mechanics [27]. The latter primarily address *dilatancy*: the tendency of dense granular media to expand upon shearing. Jamming can be viewed as a constant-volume counterpart of this process.

We await further experimental guidance on the extent to which jammed materials are, in practice, fragile. Some direct experimental tests of specific fragile models are suggested above (for jammed colloids) and elsewhere (for sandpiles) [4,7,13]. The negligibility of any incremental elastic range can be probed by various experiments including sound transmission [24]. In granular matter, computer simulations should clarify the relationship between fragility and the extreme nonlinearity arising from prohibition of tensile contact forces [22].

Meanwhile, further candidates for fragile matter include jammed colloids, weak particulate gels, and flow-induced defect textures in liquid crystals, all of which can self-organize so as to support an applied stress.

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