# Metastable State Selection in One-Dimensional Systems with a Time-Ramped Control Parameter

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The problem of state selection when multiple metastable states compete for occupation is considered for systems obeying a one-dimensional stochastic time-dependent Ginzburg-Landau equation in which a control parameter is ramped in time. The dynamics of the supercurrent in a narrow superconducting ring under the influence of an external electric field is used for illustration. [S0031-9007(98)06492-8]

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Many systems when driven far from equilibrium encounter instabilities that lead towards new states or phases. Frequently there exist multiple states that can be selected following the onset of the instability. The determination of the particular state that is selected is a complex problem of fundamental interest in a wide variety of fields [1,2]. In addition to the complexity associated with the presence of multiple competing states, if the relative stability of these states evolves in time, then the state that is selected can depend in an important way on the driving force. The focus of this paper is on state selection in such systems in one dimension.

For the purposes of this paper a ramped system is defined as one for which a control parameter is varied in time so that the system gradually progresses from stable to unstable regimes. For example, in a narrow superconducting ring [3–6] under the influence of a constant electromotive force, the superconducting electrons are accelerated by the electric field, and the supercurrent increases with time until the critical current is reached and the system becomes (Eckhaus) unstable. Similar behavior could occur in direction solidification [7–10] if the solidification cell is accelerated slowly, rather than pulled at a constant velocity, through a temperature gradient until the (Mullins-Sekerka) instability is encountered and the liquid/solid interface becomes unstable. In each of these scenarios the systems become unstable with respect to fluctuations of certain wavelengths that lie within a band. When the size of the system is comparable to the length scales associated with the wave vectors in the unstable band, the system is described as mesoscopic and the number of accessible states is finite. As illustrated in an extensive review by Cross and Hohenberg [1], instabilities that result in this type of mesoscopic behavior are extremely common, occurring in many diverse fields, such as fluid dynamics, chemical reactions, material science, and biology.

This paper focuses on such phenomena as described by a one-dimensional stochastic time-dependent Ginzburg-Landau equation. The combination of a ramped driving force and the mesoscopic system size, which allows for multiple, isolated metastable states, leads to novel and interesting selection rules. It will be shown that the rate at which the system is driven through the instability plays a prominent role in determining the probability that a particular metastable state is selected. In addition, as the decay is from states of marginal stability, the selection is also influenced by the noise in the system. The dependences on both ramping rate and noise strength are considered.

Consider an infinitely long solenoid, carrying a current that increases linearly with time, passing through the center of a narrow superconducting ring of cross-sectional area $A$ and circumference $L \equiv \xi(T)\ell$, where $\xi(T)$ is the temperature-dependent correlation length. By Faraday's law of induction, a constant electromotive force (emf) $V$ is induced in the superconductor, thereby accelerating the superconducting electrons. The dynamics of the (dimensionless) superconducting order parameter $\psi(x,t)$, where $x$ is the longitudinal spatial coordinate and $t$ is time, is described by the stochastic time-dependent Ginzburg-Landau equation [4,5],

$$\partial_t \psi = \partial_x^2 \psi + \psi - \psi|\psi|^2 + i\ell^{-1}x\omega\psi + \eta\,, \qquad (1)$$

where $\omega \equiv \tau_{\rm GL}(2eV/\hbar)$ is a dimensionless measure of the strength of the induced emf. This equation is valid for dirty superconductors near the superconducting transition in the limit that the normal current can be neglected [11]. Throughout this paper the regime where $\omega \ll 1$ is considered. In Eq. (1), $\tau_{\rm GL}$ is the Ginzburg-Landau relaxation time, and $\psi$ satisfies the twisted-periodic boundary condition $\psi(\ell + x, t) = \exp(i\omega t)\psi(x,t)$. The variable $\eta$ is a Gaussian random variable, with expectation values $\langle \eta(x,t)\rangle = 0$, and $\langle \eta(x,t)\eta^*(x',t')\rangle = 2D\delta(x - x')\delta(t - t')$, where $D$ is determined by the fluctuation-dissipation theorem [12].

For $\omega \ll 1$, the relevant current-carrying states of the superconductor are uniformly twisted plane wave solutions given by $\overline{\psi} = \sqrt{1 - q^2}\exp(iqx)$, where $q = mK + \omega t/\ell$, and $K \equiv 2\pi/\ell$. The dimensionless current density $J$ of these states is given by $J = (\overline{\psi}^*\partial_x\overline{\psi} - \overline{\psi}\partial_x\overline{\psi}^*)/2i = q(1 - q^2)$. Thus the effect of the induced emf (which increases $q$ linearly with time) is to wind the order parameter or, equivalently, to accelerate the superconducting electrons. However, this acceleration

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cannot continue indefinitely because $J$ is a nonmonotonic function of $q$, and hence time, achieving a maximum value of $J_c = 2/\sqrt{27}$ at $q = q_c = 1/\sqrt{3}$. This saturation of the current at the critical current $J_c$ coincides with the loss of stability of states $\overline{\psi}$ at $q = q_c$. In other words, for $q > q_c$, $d^2F(q)/dq^2 < 0$, where $F(q) \equiv F_{\mathrm{GL}}[\overline{\psi}]$ is the Ginzburg-Landau free energy of states $\overline{\psi}$.

To understand the Eckhaus instability for finite size systems it is necessary to perform a linear stability analysis about the state $\overline{\psi}$, as the previous analysis only applies when $\ell = \infty$. Standard linear stability analysis gives one potentially positive eigenvalue that takes the form [13]

$$\lambda_n(q) = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}\,. \quad (2)$$

The eigenvector associated with this eigenvalue is a linear combination of Fourier modes with wave vector $q \pm k_n$ and amplitude $A_n$, where $k_n = nK$. The interesting feature of this eigenvalue is that it can become positive when $q > \kappa_1 > q_c$, where $\kappa_m \equiv (1/\sqrt{3})[1 + m^2K^2/2]^{1/2}$. Thus, for finite size systems the instability is pushed to wave vectors greater than $q_c$ by an amount that depends on $\ell$ [13]. In particular, for $\kappa_m > q > \kappa_1$, $\lambda_n$ is positive for all values of $k_n < mK$ [14]. The dependence of $\lambda$ on $k_n$ is shown in the inset of Fig. 1 for several values of $q$.

The growth of a single Fourier mode (with amplitude $A_n$) of wave vector $q - k_n$, and simultaneous decay of $A_0$, corresponds to a decrease of the winding number $W = (2\pi)^{-1}\int_0^\ell d\phi(x)/dx$, where $\phi$ is the phase of $\psi$, by an amount $n$. This phenomenon is known as a "phase slip" as the total phase of the order parameter

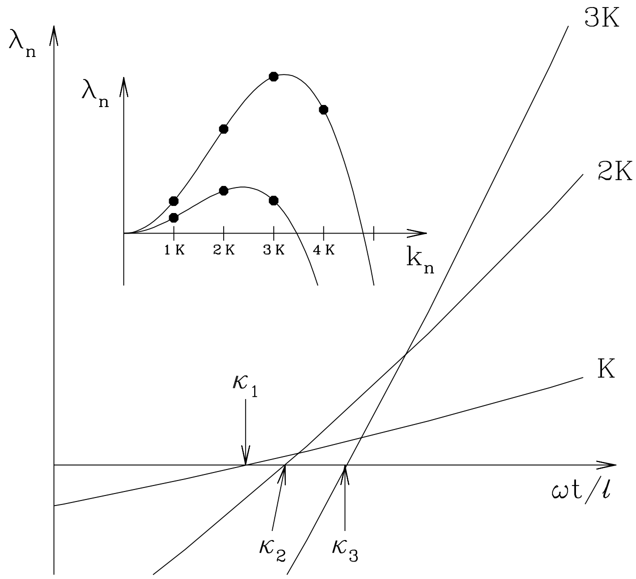


FIG. 1. $\lambda$ as a function of $q = \omega t/\ell$ for $k = K, 2K$, and $3K$. Inset: $\lambda$ as a function of $k$ for two values of $q > \kappa_1$ such that the upper curve corresponds to the larger value of $q$.

changes by an integral multiple of $2\pi$. Physically, the supercurrent decreases by a discrete amount when a phase slip occurs. Phase-slip processes can also occur via thermal activation over an energy barrier, and this process has received significant attention over the years [3,4,6,15]. For $D = 10^{-3}$, as long as $\omega \gtrsim 10^{-24}$, the probability of a thermally activated phase slip occurring is exceedingly small [5]. Thus, unless the temperature is very close to the superconducting transition temperature $T_c$, where $D$ is large, the system will almost always be driven to the Eckhaus instability before a thermally activated phase slip can occur. Therefore, the transitions that are of concern in this work involve the decay from an unstable state, in contrast to previous work [3–6] where the focus was on the decay from a metastable state.

When $\omega > 0$, the system is driven to the point of instability as the eigenvalues of each Fourier mode eventually become positive. As illustrated in Fig. 1, the $n = 1$ mode becomes unstable first, then the $n = 2$ mode becomes unstable, and so on. This implies that the system first becomes unstable with respect to single phase-slip processes, then double phase-slip processes, etc. If $\omega$ is large enough, the $n = 1$ mode might not have time to grow to dominance by the time the $n = 2$ mode becomes unstable. This suggests that for small $\omega$, single phase-slip processes should dominate the dynamics, but as $\omega$ is increased there is a crossover to a regime in which double phase-slip processes dominate. As $\omega$ is increased further, double phase-slip processes should give way to triple phase-slip processes, and so on. The generic features displayed in Fig. 1 are common to many systems and come under the general classification scheme of Cross and Hohenberg [1] as type $\mathrm{II}_s$. Thus, the dynamic competition between unstable modes discussed above is a phenomena that has relevance to many systems. For example, an analysis of the linear stability of a planar liquid/solid front in directional solidification leads to a similar structure even though Eq. (1) is not applicable to directional solidification.

To evaluate the probability of the occurrence of a given phase slip as a function of $\omega$, Eq. (1) was numerically integrated in time for a noise strength of $D = 10^{-3}$ and a length corresponding to $n_\ell \equiv \ell q_c/2\pi = 5$ [16]. In Fig. 2a, the probability of a type-$n$ phase slip is plotted as a function of $\omega$. As expected, for small $\omega$, single phase slips dominate. As $\omega$ increases further there is a crossover to a regime in which double phase slips dominate. Further increase of $\omega$ results in a subsequent crossover to a regime in which triple phase slips dominate, and so on.

An example of the dynamics that lead to such results is shown in Fig. 3. In this figure, the winding number and current are plotted as a function of time for $\omega = 5 \times 10^{-4}$. This value of $\omega$ is in the crossover region between the single and double phase-slip dominated regimes. Clearly evident in this figure are the single and double phase slips in which $W$ changes by one or two, respectively. Also seen in Fig. 3b are the discrete jumps of the supercurrent.

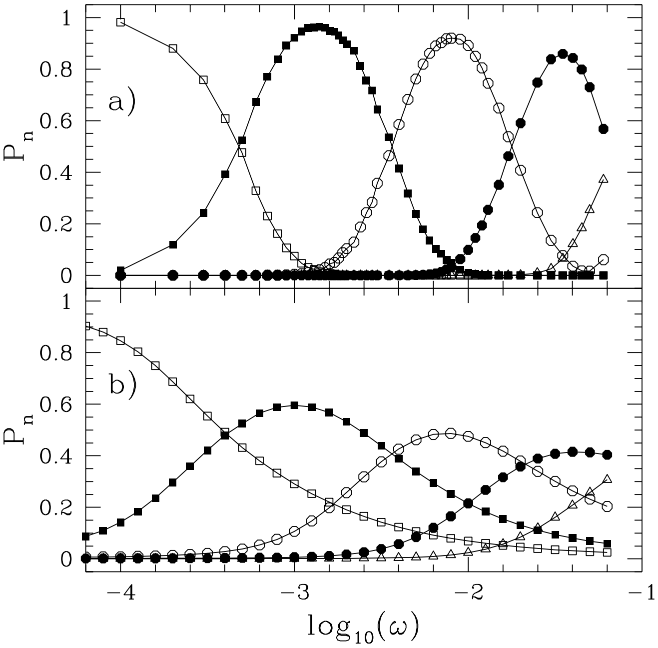


FIG. 2. State-selection probabilities as a function of the driving force $\omega$. Open squares, solid squares, open circles, solid circles, and open triangles correspond to the probabilities $P_1, P_2, P_3, P_4$, and $P_5$, respectively. Results of the numerical integration of Eq. (1) and those of the linear analysis described in the text are shown in (a) and (b), respectively.

As described earlier, the essential features shown in Fig. 2a can be understood using the properties of the growth rates $\lambda_n$. This idea can be made more concrete in the following way. Ignoring the nonlinear interactions between the different modes, the expectation of $|A_n|^2$ is given by [5]

$$\langle |A_n(t)|^2 \rangle = \frac{2D}{\ell} e^{2\sigma_n(t)} \int_0^t dt' \, e^{-2\sigma_n(t')}, \qquad (3)$$

where $\sigma_n(t) \equiv \int_0^t dt' \, \lambda_n[q(t')]$, and angular brackets denote a noise average. After the onset of the instability the system evolves towards the fixed points $\overline{\psi}_n = \bar{A}_n \times \exp[i(q - nK)x]$, where $\bar{A}_n = \sqrt{1 - (q - nK)^2}$. Equation (3) describes the initial evolution of the system after the Eckhaus boundary has been reached. In this noninteracting picture each amplitude (measured in units of $\bar{A}_n$) can be thought of as an orthogonal coordinate in an $n_\ell$-dimensional space. Thus, the natural measure of distance from the origin ($A_n = 0$) in this space is $\sum_{n=1}^{n_\ell} \langle |A_n(t)|^2 \rangle / \bar{A}_n^2$. After onset of the Eckhaus instability, this sum increases rapidly and reaches unity at time $t^*$. Assuming that at $t = t^*$ a phase slip has occurred with probability one, it is natural to interpret $\langle |A_n(t^*)|^2 \rangle / \bar{A}_n^2$ as the relative probability of the occurrence of a type-$n$ phase slip. The probabilities calculated using this procedure are shown in Fig. 2b.

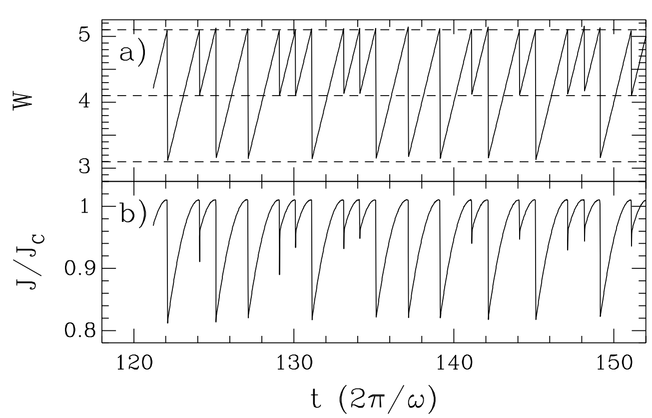


FIG. 3. Dynamics of winding number (a) and supercurrent (b), for $\omega = 5 \times 10^{-4}$ and $D = 10^{-3}$.

It is clear from Fig. 2 that the preceding analysis provides a qualitatively accurate description of the state-selection probabilities, and their dependence on the driving force $\omega$. Most notably, the values of $\omega$ at the peak positions agree very well with the numerical results. Nevertheless it is important to point out that the preceding analysis is only a plausible argument and is not systematic. A quantitative description must include the subtle nonlinear interactions that are an important element in determining state selection. Even at the present level of ignorance, however, the analysis presented here provides a qualitatively useful description of the state-selection probabilities, and their dependence on the driving force.

The growth rates $\lambda$ are an extremely important factor in determining the state-selection probabilities. The preceding analysis accounts for these growth rates and therefore provides a qualitatively accurate description. The analysis also provides predictions for the dependence of $P_n$ on the noise strength $D$, which may be more convenient to vary in some experiments. Plotted in Fig. 4a are the probabilities of a type-$n$ phase slip as a function of $D$, for a fixed value of $\omega$, obtained from a numerical simulation of Eq. (1). In Fig. 4b, the corresponding $P_n$'s obtained from the growth-rate analysis are plotted for comparison. Once again, it is seen that the simple analysis provides an accurate qualitative picture. For the smallest values of $D$ considered triple phase-slip processes dominate. This is because the time required for a given mode to grow to saturation diverges logarithmically as $D \to 0$. Consequently, if $D$ is very small, the mode amplitudes $A_1$ and $A_2$, for example, may still be very small by the time the growth rate of $A_3$ is significantly larger than the growth rates for $A_1$ or $A_2$.

One of the most interesting aspects of the phenomena exposed here is that the selection rules depend on both the intrinsic properties of the system and the external parameters. To understand this connection more deeply, it is instructive to consider the characteristic growth times for individual modes. Typically, the characteristic time associated with the initial growth of an unstable mode is taken to be the inverse of the growth rate. However, for ramped systems the growth rate $\lambda$ starts out negative and passes through zero. Thus, $|\lambda^{-1}|$ is not a relevant quantity as it diverges at the instability.

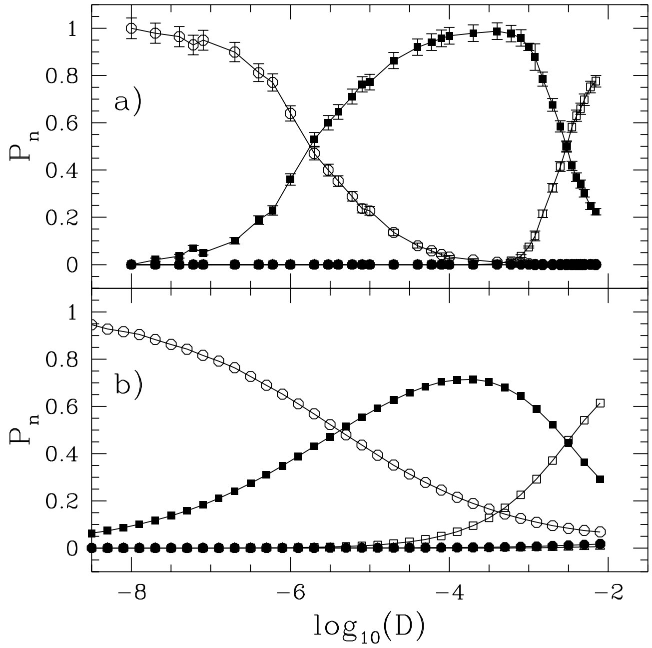


FIG. 4. State-selection probabilities as a function of the noise strength $D$. The symbols in this figure are identical to those in Fig. 2. Results of the numerical integration of Eq. (1) and those of the linear analysis described in the text are shown in (a) and (b), respectively.

To determine the characteristic time, consider Eq. (3) for $\langle |A_n(t)|^2 \rangle$. The quantity $\sigma(t)$ achieves a local minimum at $t = t_n \equiv \ell \kappa_n / \omega$ so that a second order expansion about $t_n$ yields $\sigma_n(t) \approx \sigma_n(t_n) + \frac{1}{2} \frac{\lambda_n' \omega}{\ell} (t - t_n)^2$, where $\lambda_n' \equiv \partial \lambda_n / \partial q |_{q=\kappa_n}$. Inserting this expansion into Eq. (3) and assuming that $\omega \ll 1$ gives

$$\langle |A_n(t)|^2 \rangle = 2D \tau_n \ell^{-1} \exp[z_n^2(t)] \{\mathrm{erf}[z_n(t)] + 1\}, \quad (4)$$

where $z_n(t) = (t - t_n)/\tau_n$ and $\tau_n = \sqrt{\ell / \lambda_n' \omega}$. The quantity $\tau_n$ is the characteristic time for the growth of mode $n$, and is interesting because it depends on the geometric mean of $\lambda_n'$ and $\omega$. Thus, the time scale $\tau_n$ embodies in a natural way the importance of the combination of the intrinsic dynamics ($\lambda_n'$) and the external driving force ($\omega$).

In summary, the problem of state selection in one-dimensional ramped systems described by a time-dependent Ginzburg-Landau equation has been shown to contain unique and rich phenomenology. Despite the success of the linear analysis, it is clear that new methods must be developed to explore this complex and important area of research in nonequilibrium statistical mechanics. Recent work [17] on state selection in non-ramped marginally stable systems suggests a possible systematic framework that could be extended to address the phenomena considered here.

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