Quantized Vortices and Collective Oscillations of a Trapped Bose-Einstein Condensate

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Using a sum rule approach we calculate the frequency shifts of the quadrupole oscillations of a harmonically trapped Bose gas due to the presence of a quantized vortex. Analytic results are obtained for positive scattering lengths and large N where the shift relative to excitations of opposite angular momentum is found to be proportional to the quantum circulation of the vortex and to decrease as $N^{-2/5}$. Results are also given for smaller values of N covering the transition between the ideal gas and the Thomas-Fermi limit. The splitting of the collective frequencies in toroidal configurations is also discussed. [S0031-9007(98)06849-5]

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After the experimental realization of Bose-Einstein condensation in dilute atomic gases, the study of the collective excitations of these unique inhomogeneous quantum systems has been the object of both experimental [1–4] and theoretical [5,6] work (for a recent theoretical review, see [7]). These oscillations are characterized by proper quantum numbers, reflecting the symmetry of the confining potential. In an axially symmetric trap the third component of angular momentum is a natural quantum number and if the system is in a time reversal invariant configuration, elementary excitations carrying opposite angular momentum are degenerate. This degeneracy is in general removed if time reversal symmetry is broken. The purpose of this Letter is to describe the frequency shifts produced by the presence of a quantized vortex. In view of the important role played by vortices in understanding the mechanisms of superfluidity, the possibility of their spectroscopic diagnostics is highly interesting since the measurements of collective frequencies can be carried out with high precision in these systems.

The occurrence of splitting in the presence of a vortex can be simply understood by noting that the average velocity flow associated with the collective oscillation can be either parallel or opposite to the vortex flow, depending on the sign of the angular momentum carried by the excitation. This produces a shift of the collective frequency of order $\delta\omega/\omega \sim v/c$, where $v \sim 1/R$ is the velocity of the vortex flow and c is the sound velocity. In a trapped Bose gas c increases linearly with the radius R of the condensate, while ω is practically independent of R, so one expects relative shifts of the order of $1/R^2$. These effects are larger than the typical corrections to the Thomas-Fermi limit due to finite size effects which, in the absence of the vortex, behave like $\log R/R^4$ [8]. The problem of the frequency shift produced by a quantized vortex has been already the object of theoretical work using semiclassical approaches based on a large N expansion [9], as well as by full numerical solution of the linearized equations of motion [10].

In this Letter we develop a sum rule approach [11] which is expected to provide exact results for the frequency splitting in large systems. The method can be also applied to calculate the frequency shifts for small values of N. Let us introduce the strength distribution function

$$S_{\pm}(E) = \sum_n |\langle n|F_{\pm}|0\rangle|^2 \delta(E - \hbar\omega_{n0}) \tag{1}$$

relative to the operators $F_{\pm} = \sum_{k=1}^N f_{\pm}(\mathbf{r}_k)$ carrying opposite angular momenta. In Eq. (1) $\hbar\omega_{n0} = (E_n - E_0)$ are the excitation energies relative to the eigenstates $|n\rangle$ of the Hamiltonian

$$H = \sum_i \left(\frac{1}{2M} p_i^2 + V_{\mathrm{ext}}(\mathbf{r}_i)\right) + g \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j), \tag{2}$$

which describes N interacting bosons confined by an external potential. The potential $V_{\mathrm{ext}}(\mathbf{r}) = M(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2)/2$, with $r_\perp^2 = x^2 + y^2$, is assumed to be axially symmetric, and the interatomic force is a contact two-body interaction whose coupling constant $g = 4\pi\hbar^2 a/M$ is fixed by the s-wave scattering length a.

In the following we will focus on the collective oscillations of low multipolarity which are easily excited in experiments by suitable modulation of the harmonic trap. For the quadrupole case we will consider the modes excited by the operators

$$f_{\pm} = (x \pm iy)^2 \tag{3}$$

and

$$f_{\pm} = (x \pm iy) z, \tag{4}$$

carrying angular momentum $m = \pm 2$ and $m = \pm 1$, respectively (here and in the following we will identify m with the third component of angular momentum of the elementary excitation). Only excitations with $m \neq 0$ are relevant for the present discussion.

In the absence of vortices the ground state has zero angular momentum and, for large N and positive scattering lengths, the collective states excited by the operators (3) and (4) are well described by hydrodynamic theory of superfluids. This yields [6] the result $\omega_{\pm} = \sqrt{2}\,\omega_\perp$ and $\omega_{\pm} = \sqrt{\omega_\perp^2 + \omega_z^2}$ for the $m = \pm 2$ and $m = \pm 1$

frequencies. Notice that these results differ from the ideal gas predictions, $\omega_\pm = 2\omega_\perp$ and $\omega_\pm = \omega_\perp + \omega_z$, as a consequence of interaction effects which suppress the contribution of the kinetic energy pressure term in the equations of motion. Differently from the quadrupole excitations, the frequencies of the dipole modes excited by $f_\pm = x \pm iy$ are instead unaffected by two-body interactions and are given by $\omega_\pm = \omega_\perp$. This behavior is the consequence of the translational invariance of the interatomic force which cannot affect the motion of the center of mass, even in the presence of a vortex.

The moments

$$m_p^\pm = \int_0^\infty dE\,[S_+(E) \pm S_-(E)]\,E^p \qquad (5)$$

of the strength distribution (1) can be calculated using closure relations. For the lowest moments we find the following results:

$$m_0^- = \langle [F_-, F_+] \rangle = 0\,, \qquad (6)$$

$$m_1^+ = \langle [F_-, [H, F_+]] \rangle = \frac{N\hbar^2}{M} \langle |\nabla f_+|^2 \rangle\,, \qquad (7)$$

$$m_2^- = \langle [[F_-, H], [H, F_+]] \rangle = N\langle [j_-, j_+] \rangle\,, \qquad (8)$$

where the average $\langle\,\rangle$ is taken on the state $|0\rangle$ which may or may not contain a vortex and we have used the property $F_+^\dagger = F_-$.

The first commutator (6) vanishes because the operators F_+ and F_- depend only on the spatial coordinates. The double commutator (7) is the analog of the f-sum rule [12] and gets contributions only from the kinetic energy term since both the external potential and the two-body interaction commute with $F_\pm$. Finally the current operators entering the third sum rule are defined by

$$j_\pm = \frac{\hbar}{2Mi} \nabla f_\pm(\mathbf{r}) \cdot \mathbf{p} + \text{H.c.}\,, \qquad (9)$$

where $\mathbf{p}$ is the usual momentum operator. Evaluation of the sum rules m_1^+ and m_2^- is straightforward in the case of the quadrupole operators (3) and (4). For $m = \pm 2$ we find the result

$$m_1^+ = \frac{8\hbar^2}{M} N\langle r_\perp^2 \rangle\,, \qquad (10)$$

$$m_2^- = \frac{16\hbar^3}{M^2} N\langle l_z \rangle = \frac{16\hbar^4}{M^2} N\kappa\,, \qquad (11)$$

where l_z is the zth component of the angular momentum operator, whose average value, in the case of axially symmetric geometry, can be expressed in terms of the quantum of circulation, κ, of the vortex ($\kappa = \pm 1, \pm 2, \ldots$).

The results for the moments m_0^-, m_1^+, and m_2^- can be used to calculate the shift of the collective frequencies when the number of atoms in the trap is large and a is positive. In fact, in this limit, where the behavior of the system is properly described by hydrodynamic theory of superfluids, one expects that the moments calculated above will be exhausted by two modes with frequency $\omega_\pm$ excited, respectively, by the operators $F_\pm$:

$$S_\pm(E) = \sigma^\pm \delta(E - \hbar\omega_\pm)\,, \qquad (12)$$

where $\sigma^\pm$ are the corresponding strengths. Assumption (12) is equivalent to a Bijl-Feynman ansatz [13] often used to describe the collective excitations in interacting many-body systems. In the case of superfluid helium it provides an exact description of the excitation spectrum in the phonon regime (for a recent discussion on sum rules and collective excitations in Bose superfluids, see, for example, [14]).

Let us discuss the consequence of the vanishing of the m_0^- moment (6). With assumption (12) for the strength distribution one immediately finds the result $\sigma^+ = \sigma^-$ and the splitting between the two frequencies can be directly written as

$$\hbar(\omega_+ - \omega_-) = m_2^-/m_1^+\,. \qquad (13)$$

Use of (10) and (11) then yields the relevant result

$$\omega_+ - \omega_- = \frac{2}{M}\frac{\langle l_z \rangle}{\langle r_\perp^2 \rangle} = \frac{7\omega_\perp \kappa}{\lambda^{2/5}} \left(15 \frac{Na}{a_\perp}\right)^{-2/5} \qquad (14)$$

for the $m = \pm 2$ modes, where $a_\perp = \sqrt{\hbar/M\omega_\perp}$ is the oscillator length in the radial direction, while $\lambda = \omega_z/\omega_\perp$ characterizes the deformation of the harmonic trap. The same calculation can be carried out for the modes excited by the $m = \pm 1$ quadrupole operators (4). In this case the result is

$$\omega_+ - \omega_- = \frac{2}{M}\frac{\langle l_z \rangle}{\langle r_\perp^2 + 2z^2 \rangle} = \frac{7\omega_\perp \kappa \lambda^{8/5}}{1 + \lambda^2} \left(15 \frac{Na}{a_\perp}\right)^{-2/5}.$$

In the last equalities of the above equations we have used the Thomas-Fermi approximation [15] to evaluate the square radii of the condensate. Notice that for spherical trapping the splitting between the $m = \pm 2$ quadrupole frequencies is twice the splitting between the $m = \pm 1$ modes. It is worth noting that the above results depend on macroscopic features of the system (angular momentum and square radii) and for large N they are consequently insensitive to microscopic detalis of the wave function, such as, for example, the structure of the vortex core. For large N the shifts become smaller and smaller showing that in this limit the effects associated with the current of the vortex are small corrections to the collective flow of the oscillation. Nevertheless the splittings can be sizable. For example, using a spherical configuration with $a/a_\perp = 10^{-3}$ and $N = 10^6$ and taking one quantum of circulation ($\kappa = 1$) one finds that the relative shift $(\omega_+ - \omega_-)/\omega$ of the $m = \pm 2$ states is about 10%, having used the large N result $\sqrt{2}\,\omega_\perp$ for the average frequency ω. This shift is much larger than the typical experimental uncertainties in the measurements of the collective frequencies [1–4].

It is worth pointing out that the frequency shift due to the vortex is exhibited by the quadrupole excitations, but not by the dipole modes excited by the operators $f_\pm = x \pm iy$. In fact in the dipole case the operators $j_\pm = \hbar(p_x \pm ip_y)/Mi$ commute and the sum rule m_2^- identically vanishes. As a consequence one finds $\omega_+ = \omega_-$ as expected from general arguments. Notice, however, that in addition to the above modes excited by the center of mass operators, another dipole mode, localized near the core of the vortex, has been predicted [10] to occur with frequency different from $\omega_\perp$. It has been suggested [16] that this mode could play an important role in driving the instability of the vortex.

The above results for the shift of the quadrupole frequencies hold for large N, where the assumption (12) that the operators f_+ and f_- excite a single mode is justified. When the dimensionless parameter $Na/a_\perp$ becomes small, this assumption is no longer valid. In particular, in the limit of a noninteracting gas, a vortex with quantum of circulation $\kappa = +1$ corresponds to putting all the atoms in the $1p$ state ($l_z = +1$) of the harmonic oscillator hamiltonian and the $m = -2$ operator f_- gives rise to $\delta\omega = 2\omega_0$ as well as to $\delta\omega = 0\omega_0$ transitions, and, vice versa, the operator f_+ gives rise only to $\delta\omega = 2\omega_0$ transitions. The corresponding strengths are, respectively, $\sigma_{\text{up}}^- = 2a_\perp^4 N$, $\sigma_{\text{down}}^- = 4a_\perp^4 N$, and $\sigma^+ = 6a_\perp^4 N$. As a consequence, in order to study the transition from the noninteracting to the large N regime we have to remove the single-mode assumption (12) for the strength distribution S_-. In the following we will use the ansatz $S_-(E) = \sigma_{\text{up}}^- \delta(E - \hbar\omega_{\text{up}}^-) + \sigma_{\text{down}}^- \delta(E - \hbar\omega_{\text{down}}^-)$ which properly accounts for the behavior of the strength distribution in both the large and small N limits. Of course in this case the knowledge of the three sum rules (6)–(8) is no longer sufficient to determine the collective frequencies and result (13) for the splitting is no longer valid. In order to calculate the new frequencies and strengths we have evaluated the additional moments m_3^+, m_4^-, and m_5^+. For the $m = \pm 2$ quadrupole operators (3) we find the results

$$m_3^+ = \frac{16\hbar^4\omega_\perp^2}{M} N\langle r_\perp^2\rangle \left[1 + \frac{E_{\text{kin}_\perp}}{E_{\text{ho}_\perp}}\right], \tag{15}$$

$$m_4^- = \frac{64\hbar^5\omega_\perp^2}{M^2} N\langle l_z\rangle = \frac{64\hbar^6\omega_\perp^2}{M^2} N\kappa, \tag{16}$$

$$m_5^+ = \frac{32\hbar^6\omega_\perp^4}{M} N\langle r_\perp^2\rangle \left[1 + 3\frac{E_{\text{kin}_\perp}}{E_{\text{ho}_\perp}} + \frac{\tilde{V}_{\text{int}}}{16E_{\text{ho}_\perp}}\right]. \tag{17}$$

In (15)–(17) $E_{\text{kin}_\perp}$ and $E_{\text{ho}_\perp}$ are the radial contributions to the kinetic and oscillator energies, respectively, and

$$\tilde{V}_{\text{int}} = g a_\perp^4 \int d\mathbf{r}\left[\frac{8\kappa^2 n_0^2}{r_\perp^4} + \nabla_\perp^2 n_0 \left(\frac{|\nabla_\perp n_0|^2}{n_0} - \nabla_\perp^2 n_0\right)\right]$$

is the contribution to m_5^+ arising from two-body interactions where n_0 is the density of the condensate and κ is the quantum of circulation of the vortex. Result (15) for m_3^+ permits one to obtain directly the large N behavior of the quadrupole frequency. In fact, in this limit the kinetic energy term in (15) is negligible and the ratio $(m_3^+/m_1^+)^{1/2}$ approaches the hydrodynamic result $\sqrt{2}\,\hbar\omega_\perp$. Result (16) for m_4^- provides a crucial check of the validity of the main result (14). In fact, one can see that in the large N limit the single-mode approximation (12), with the dispersion law

$$\omega_\pm = \sqrt{2}\,\omega_\perp \pm \Delta\omega, \tag{18}$$

and $\Delta\omega = \omega_+ - \omega_-$ given by (14), is consistent with result (16) for the sum rule m_4^+, up to effects linear in $\Delta\omega$. Differently from the other sum rules, m_5^+ depends explicitly on the two-body interaction. This contribution is very sensitive to the core region of the vortex and is important to describe the crossover from the noninteracting to the Thomas-Fermi limit.

We have calculated numerically the sum rules $m_p^\pm$ with $p = 1, \ldots, 5$ using the solution of the Gross-Pitaevskii equation for a vortex of quantum circulation $\kappa = +1$ [17]. These sum rules are then used to evaluate the quadrupole energies and strengths using the new ansatz for the strength distribution. The results for a spherical potential are reported in Fig. 1, where, for simplicity, we have plotted only the frequency ω_{down}^- of the lowest mode excited by f_-. One can prove that ω_{down}^-, calculated with the above procedure, corresponds to a rigorous upper bound to the lowest frequency of the modes given by the full solution of the linearized Gross-Pitaevskii equation. The strength relative to the high frequency mode excited by f_- vanishes rapidly when N increases and hence this mode is not physically relevant, except for small values of N. In the figure we also report the average energy $\hbar\omega_{3,1} = (m_3^+/m_1^+)^{1/2}$ calculated in the absence of vortices. This energy turns out to be very close to the exact numerical solution of the linearized Gross-Pitaevskii equation (differences are always smaller than 0.5%).

The frequencies $\omega_\pm$ start, respectively, from $2\omega_\perp$ and $0\omega_\perp$ and approach the value $\sqrt{2}\,\omega_\perp$ for large N. In the same limit the strengths σ^+ and σ_{down}^- tend to the asymptotic value $4\sqrt{2}\,a_\perp^4 N\lambda^{2/5}(15Na/a_\perp)^{2/5}/7$. The figure shows that for large $Na/a_\perp$ the asymptotic dispersion relation (18) reproduces well the behavior of the two collective frequencies. Notice that for such values of $Na/a_\perp$ the strengths σ^+ and σ_{down}^- practically coincide, confirming the validity of the single-mode assumption (12). The strength σ_{up}^- is simply given by the difference $\sigma^+ - \sigma_{\text{down}}^-$ and differs significantly from zero only for very small values of $Na/a_\perp$. The small asymmetry of the calculated frequencies with respect to the hydrodynamic value is a consequence of the fact that for the values of $Na/a_\perp$ reported in the figure the excitation energy differs from $\sqrt{2}\,\omega_\perp$ even in the absence of vortices.

We have also explored in an explicit way the case of weak interactions ($Na/a_\perp \ll 1$): in this case the

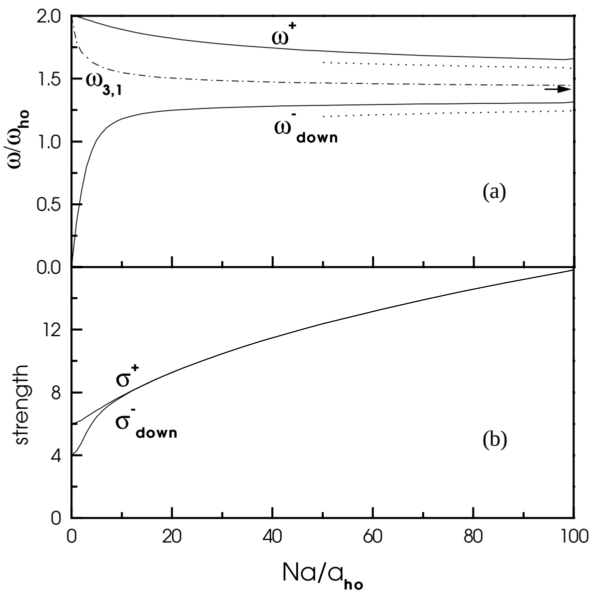

FIG. 1. Frequencies (a) and strengths (b) relative to the $m = \pm 2$ quadrupole modes in the presence of a $\kappa = 1$ vortex, as a function of $Na/a_\perp$ for a spherical trap. The dotted lines correspond to the large N behavior (18). The arrow indicates the Thomas-Fermi limit $\omega = \sqrt{2}\omega_\perp$. The dashed dotted line corresponds to the ratio $(m_3^+/m_1^+)^{1/2}$ without vortex. Strengths are given in units of $a_\perp^4 N$.

dispersion law of the lowest energy solution is given by

$$\omega^-_{\rm down} = \omega_\perp \left(\frac{Na}{a_\perp}\right)\sqrt{\frac{\lambda}{2\pi}}\,. \qquad (19)$$

The excitation energy (19) becomes negative for $a < 0$. It has been, however, shown that systems interacting with attractive forces do not exhibit vortical configurations because of the fragmentation of the condensate [18].

It is finally interesting to discuss how the above results are modified by changing the geometry of the problem. This is important because toroidal configurations are expected to suppress the mechanisms of instability of the vortex [16]. A useful way to pin a vortex might be achieved with a thin laser beam stabilizing the core of the vortex along the z axis. The thickness d of the beam can be a few microns. This is larger than the size of the core, fixed by the coherence length $\xi = a_\perp^2/R$, but can be significantly smaller than the size of the condensate R. In this case the structure of the core of the vortex will be significantly modified by the pinning, but the macroscopic behavior of the collective excitations and, in particular, result (14) for the splitting, will be modified in a minor way. An estimate of this effect can be obtained by calculating the change of $\langle r_\perp^2\rangle$ due to the presence of the repulsive potential generated by the laser beam. We expect small effects if $d \ll R$.

A quite different behavior is achieved by choosing a ring geometry. In the case of an ideal ring of radius R the problem is analytically soluble using Bogoliubov theory [16]. In this case the natural excitation operators have the form $f_\pm = \exp(\pm im\phi)$, where ϕ is the azimuthal angle and $\pm m$ is the angular momentum carried by the excitation. The kinetic energy operator can be replaced by $(-\hbar^2/2MR^2)\partial^2/\partial\phi^2$ and the sum rules (7) and (8) become $m_1^+ = \hbar^2 m^2/MR^2$ and $m_2^- = 2\hbar^3 m^3\langle l_z\rangle/M^2R^4$. Using the Bijl-Feynman ansatz (12) one immediately finds $\omega_+ - \omega_- = 2\hbar m\kappa/MR^2$, where κ is the quantum of circulation of the vortex, in agreement with the results recently discussed in [16]. Notice that for large R the lowest collective excitations in the ring geometry correspond to one-dimensional compression waves with dispersion

$$\omega = c|m|/R \pm \hbar\kappa m/MR^2\,, \qquad (20)$$

where c is the sound velocity. These frequencies, which are the analog of (18), coincide with the ones calculated for a system at rest in a frame rotating with angular velocity $\omega = \hbar\kappa/MR^2$.

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